Lecture 34 Wed Nov 20

Above corresp gives bijection between

(i) \( B(k_1) \oplus \cdots \oplus B(k_l) \oplus B(k_r) \)

(ii) the set of \( k \times n \) matrices with entries in \( \mathbb{N} \) and row sums \( b_1, b_2, \ldots, b_k \)

Define crystal \( \mathcal{B} = \mathcal{B}(k,n) \) by

\[
\mathcal{B} = \bigcup_{b_1, b_2, \ldots, b_k \in \mathbb{N}} B(k_1) \oplus \cdots \oplus B(k_l) \oplus B(k_r)
\]

Above bijection induces a bijection between

(i) \( \mathcal{B}(k,n) \)

(ii) the set of \( k \times n \)-matrices with entries in \( \mathbb{N} \)
Next we find the hw elements in $\mathcal{B}(k\lambda)$.

Recall the crystal

$$\mathcal{B}(\lambda_1) \otimes \cdots \otimes \mathcal{B}(\lambda_k) \otimes \mathcal{B}(\lambda_k)$$

Fix a partition $\lambda \in \Lambda^+$ we get a bijection between

(i) set of hw elements $x \in \mathfrak{A}$ with wt $\lambda$

(ii) set of semi standard tableaux $Q$ of

shape $\lambda$ with entries in $\{1, 2, \ldots, k\}$, such that exactly

- $k_1$ boxes of $Q$ contain 1
- $k_2$ boxes of $Q$ contain 2
- $\ldots$
- $k_k$ boxes of $Q$ contain $k$

Fix $x \in \mathfrak{A}$ describe corresponding $Q$

Write $x = x_1 \otimes \cdots \otimes x_2 \otimes x_k$
For $i \leq k$, the entries in $X_i$ describe the locations in $Q$ that contain $i$.

For $i > k$, the entries in $X_i$ that contain $j$:

- # boxes in row $j$ of $Q$ that contain $i$
\( x \times n = 3 \quad k = 4 \)

\[ X = \begin{array}{ccc} 2 & 2 & 3 \\ \end{array} \otimes \begin{array}{ccc} 1 & 1 & 2 & 3 \\ \end{array} \otimes \begin{array}{ccc} 1 & 2 \\ \end{array} \otimes \begin{array}{ccc} 1 & 1 \\ \end{array} \]

\[ Q = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \\ 3 & 4 \\ \end{array} \]

\[ \text{Obs} \]

\[ \text{skip} \quad Q = (5, 4, 2) = \text{wt} (x) \]

\[ \text{check} \quad X \text{ is } hw \]

\[ \text{Apply signature rule} \quad \]

\[ 2 \ 2 \ 3 \ \otimes \ \begin{array}{ccc} 1 & 1 & 2 & 3 \\ \end{array} \ \otimes \begin{array}{ccc} 1 & 2 \\ \end{array} \ \otimes \begin{array}{ccc} 1 & 1 \\ \end{array} \]

\[ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \]

\[ \text{reduced to} \quad \]

\[ \text{by} \]

\[ \psi_1 (x) = 1, \quad \varepsilon_1 (x) = 0 \]

\[ \uparrow \]
For a partition $\lambda \in \Lambda^+$, above bijection induces a bijection between

(i) the set of hook elements $x \in \mathbb{B}(k,n)$ that have wt $\lambda$

(ii) the set of semi-standard tableaux $\mathcal{P}$ of shape $\lambda$

that have entries in $\{1, 2, \ldots, k\}$

We are now ready to describe the extended RSK.
Extended RSK gives a bijection between

(i) The set of $k \times n$ matrices with entries in $\mathbb{N}$

(ii) The set of ordered pairs $(P, Q)$ such that

$P = \text{s. std. tableau with entries in } \{1, 2, \ldots, n\}$
$Q = \text{s. std. tableau with entries in } \{1, 2, \ldots, k\}$

$P, Q$ have same shape.

For $x \in (i)$ we now describe corresponding $(P, Q) \in (ii)$

We may identify $x$ with an element in $B(k,n)$

$x \in$ connected component $C$ of $B(k,n)$

$\exists$ partition $\lambda \in \Lambda^+$ st

$C$ is $\mathcal{B}_\lambda$

hence crystal is $\mathcal{B}_\lambda$

$C \rightarrow \mathcal{B}_\lambda$

defines $x \rightarrow P$
By construction,

\[ P = \text{s. standard tableau of shape } \lambda \]

with entries in \([1, 2, \ldots, n]\)

Also,

\[ C \text{ has unique hw element (collect } x) \]

\[ \bar{x} \text{ has hw } \lambda \]

So \( \bar{x} \) corresponds to a s. standard tableau \( \varphi \)

of shape \( \lambda \) with entries in \([1, 2, \ldots, k]\).

\( P, \varphi \) have same shape \( \lambda \).

We have obtained \((P, \varphi)\)
Next goal: describe extended RSK using Schensted insertion

Start with \( kn \times m \) matrix \( A \) with entries in \( \mathbb{N} \)

\[ k = 3, \quad n = 4 \]

\[ A = \begin{pmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix} \]

Consider a bipartite graph with vertices

\[ 1 \quad 2 \quad \vdots \quad k \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad h \]

For \( 1 \leq i, k \) and \( 1 \leq j, h \) the graph has

\( A_{ij} \) edges between \( i \) and \( j \)

\[ \begin{array}{ccccc}
& 1 & & & 1 \\
1 & & & & \\
2 & & & & \\
\vdots & & & & \\
k & & & & \\
h & & & & \\
\end{array} \]
For above graph, represent each edge by column vector

\[
\begin{bmatrix}
i \\ j
\end{bmatrix}
\]

List the resulting column vectors in \text{LEX} order (viewing \( [i] \) as a word \( i \)) to get \( A \) in 2-line notation. \( * \) yields

\[
\begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 2 & 2 & 4 & 2 & 3 & 3 & 1 & 2 & 3
\end{bmatrix}
\]

Next apply Schensted Insertion to 2nd row, in 11 stages suggested by 1st row
<table>
<thead>
<tr>
<th>Grade</th>
<th>Insehrm sequence</th>
<th>record shapes</th>
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<tbody>
<tr>
<td>1</td>
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</table>

*Note: The shapes in the record column are not fully described.*

*Note: The sequence of shapes in the recording tableau is not fully described.*
By construction

\[ P = \text{some \ stand \ tableau \ with \ entries \ in \ } \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r \]

show

\[ Q = \text{some \ stand \ tableau \ with \ entries \ in \ } \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r \]

This part of

By extension

\[ P \cup Q \text{ same step} \]