Lowering-Raising triples and $U_q(sl_2)$

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We introduce the notion of a **Lowering-Raising triple** (or **LR triple**) of linear transformations.

We will classify up to equivalence the LR triples over an algebraically closed field.

The solutions fall into six infinite families.

We will show how each family is linked to $U_q(sl_2)$ or a related algebra.

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**Lowering-Raising triples and $U_q(sl_2)$**
I Statement of the problem
II Linear algebra preliminaries
III Toeplitz matrices
IV LR pairs
V LR triples
VI The classification of LR triples
Let $\mathbb{F}$ denote a field.

Fix an integer $d \geq 0$.

Let $V$ denote a vector space over $\mathbb{F}$ with dimension $d + 1$. 
By a **decomposition of** $V$ we mean a sequence $\{ V_i \}_{i=0}^{d}$ of one dimensional subspaces whose direct sum is $V$.

We represent this decomposition by a sequence of dots:

$$
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
V_0 & V_1 & \cdots & \cdots & V_d \\
\end{array}
$$
Lowering and Raising maps

Let \( \{ V_i \}_{i=0}^{d} \) denote a decomposition of \( V \).

Consider a linear transformation \( A \in \text{End}(V) \).

We say that \( A \) lowers \( \{ V_i \}_{i=0}^{d} \) whenever \( AV_i = V_{i-1} \) for \( 1 \leq i \leq d \) and \( AV_0 = 0 \).

We say that \( A \) raises \( \{ V_i \}_{i=0}^{d} \) whenever \( AV_i = V_{i+1} \) for \( 0 \leq i \leq d-1 \) and \( AV_d = 0 \).
Lowering and Raising maps, cont.

$\leftarrow A$ lowers

$\cdots$

$A$ raises $\rightarrow$

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LR pairs

An ordered pair $A, B$ of elements in $\text{End}(V)$ is called **Lowering-Raising** (or **LR**) whenever there exists a decomposition of $V$ that is lowered by $A$ and raised by $B$.

We refer to such a pair as an **LR pair on** $V$.

This LR pair is said to be over $\mathbb{F}$.

We call $d$ the **diameter** of the pair.
An LR triple on $V$ is a sequence $A, B, C$ of elements in $\text{End}(V)$ such that any two of $A, B, C$ form an LR pair on $V$.

This LR triple is said to be over $\mathbb{F}$.

We call $d$ the diameter of the triple.
Let $A, B$ denote an LR pair on $V$. Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $d + 1$, and let $A', B'$ denote an LR pair on $V'$. By an isomorphism of LR pairs from $A, B$ to $A', B'$ we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ such that $\sigma A = A'\sigma$ and $\sigma B = B'\sigma$. The LR pairs $A, B$ and $A', B'$ are called isomorphic whenever there exists an isomorphism of LR pairs from $A, B$ to $A', B'$.

Isomorphism for LR triples is similarly defined.
Let $A, B$ denote an LR pair over $\mathbb{F}$.

Let $\alpha, \beta$ denote nonzero scalars in $\mathbb{F}$.

Then the pair $\alpha A, \beta B$ is an LR pair over $\mathbb{F}$.

An LR pair $A', B'$ over $\mathbb{F}$ is said to be equivalent to $A, B$ whenever there exist nonzero $\alpha, \beta \in \mathbb{F}$ such that $A', B'$ is isomorphic to $\alpha A, \beta B$.

Equivalence of LR triples is similarly defined.
**Problem:** Classify up to equivalence the LR triples over $\mathbb{F}$.

In this talk we give the solution, under the assumption that $\mathbb{F}$ is algebraically closed.
Fix a vector space $V$ over $\mathbb{F}$ of dimension $d + 1$.

Let $\{v_i\}_{i=0}^{d}$ denote a basis for $V$. For $0 \leq i \leq d$ let $V_i$ denote the span of $v_i$.

Then the sequence $\{V_i\}_{i=0}^{d}$ is a decomposition of $V$, said to be 
**induced** by the basis $\{v_i\}_{i=0}^{d}$.
By a **flag on** $V$ we mean a sequence $\{U_i\}_{i=0}^{d}$ of subspaces of $V$ such that $U_i$ has dimension $i + 1$ for $0 \leq i \leq d$ and $U_{i-1} \subseteq U_i$ for $1 \leq i \leq d$.

Let $\{V_i\}_{i=0}^{d}$ denote a decomposition of $V$. For $0 \leq i \leq d$ define $U_i = V_0 + \cdots + V_i$.

Then the sequence $\{U_i\}_{i=0}^{d}$ is a flag on $V$.

This flag is said to be **induced** by the decomposition $\{V_i\}_{i=0}^{d}$. 

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Flags and bases

Let \( \{ u_i \}_{i=0}^d \) denote a basis of \( V \).

This basis induces a decomposition of \( V \), which in turn induces a flag on \( V \).

This flag is said to be **induced** by the basis \( \{ u_i \}_{i=0}^d \).

Let \( \{ u_i \}_{i=0}^d \) and \( \{ v_i \}_{i=0}^d \) denote bases of \( V \). Then the following are equivalent:

(i) the transition matrix from \( \{ u_i \}_{i=0}^d \) to \( \{ v_i \}_{i=0}^d \) is upper triangular;

(ii) \( \{ u_i \}_{i=0}^d \) and \( \{ v_i \}_{i=0}^d \) induce the same flag on \( V \).
Suppose we are given two flags on $V$, denoted $\{U_i\}_{i=0}^{d}$ and $\{U'_i\}_{i=0}^{d}$.

These flags are called **opposite** whenever there exists a decomposition $\{V_i\}_{i=0}^{d}$ of $V$ that induces $\{U_i\}_{i=0}^{d}$ and whose inversion $\{V_{d-i}\}_{i=0}^{d}$ induces $\{U'_i\}_{i=0}^{d}$.

In this case $V_i = U_i \cap U'_{d-i}$ for $0 \leq i \leq d$. 
Let \( \{ V_i \}_{i=0}^d \) denote a decomposition of \( V \) that is lowered by \( A \in \text{End}(V) \).

Then \( V_i = A^{d-i} V_d \) for \( 0 \leq i \leq d \). Moreover \( A^{d+1} = 0 \).

For \( 0 \leq i \leq d \) the subspace \( V_0 + \cdots + V_i \) is the kernel of \( A^{i+1} \) and equal to \( A^{d-i} V \).

The sequences \( \{ \ker A^{i+1} \}_{i=0}^d \) and \( \{ A^{d-i} V \}_{i=0}^d \) both equal the flag on \( V \) induced by \( \{ V_i \}_{i=0}^d \).
An element $A \in \text{End}(V)$ will be called \textbf{Nil} whenever $A^{d+1} = 0$ and $A^d \neq 0$. 
Lemma

For $A \in \text{End}(V)$ the following are equivalent:

(i) $A$ is Nil;

(ii) there exists a decomposition of $V$ that is lowered by $A$;

(iii) there exists a decomposition of $V$ that is raised by $A$;

(iv) for $0 \leq i \leq d$ the kernel of $A^{i+1}$ is $A^{d-i}V$;

(v) the kernel of $A$ is $A^dV$;

(vi) the sequence $\{\ker A^{i+1}\}_{i=0}^d$ is a flag on $V$. 
Let $\{\alpha_i\}_{i=0}^d$ denote scalars in $\mathbb{F}$.

Let $T$ denote an upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$.

Then $T$ is said to **Toeplitz, with parameters** $\{\alpha_i\}_{i=0}^d$ whenever $T$ has $(i,j)$-entry $\alpha_{j-i}$ for $0 \leq i \leq j \leq d$. 
In this case

$$T = \begin{pmatrix}
\alpha_0 & \alpha_1 & \cdots & \cdots & \alpha_d \\
\alpha_0 & \alpha_1 & \cdots & \cdots \\
\alpha_0 & \cdots & \cdots \\
0 & \cdots & \cdots & \alpha_1 \\
\alpha_0 & \alpha_1 \\
\end{pmatrix}. $$
The above Toeplitz matrix $T$ is invertible if and only if $\alpha_0 \neq 0$.

In this case, $T^{-1}$ is upper triangular and Toeplitz:

$$T^{-1} = \begin{pmatrix}
    \beta_0 & \beta_1 & \cdots & \cdots & \beta_d \\
    \beta_0 & \beta_1 & \cdots & \cdots & \\
    \beta_0 & \cdots & \cdots & \cdots & \\
    \cdots & \cdots & \cdots & \beta_1 & \\
    0 & \cdots & \cdots & \beta_1 & \beta_0
\end{pmatrix}.$$
Assuming $\alpha_0 = 1$ we have

$$
\begin{align*}
\beta_0 &= 1, \\
\beta_1 &= -\alpha_1, \\
\beta_2 &= \alpha_1^2 - \alpha_2 \\
\beta_3 &= 2\alpha_1\alpha_2 - \alpha_1^3 - \alpha_3 \\
\beta_4 &= \alpha_1^4 + 2\alpha_1\alpha_3 + \alpha_2^2 - 3\alpha_1^2\alpha_2 - \alpha_4
\end{align*}
$$

(If $d \geq 1$),

(If $d \geq 2$),

(If $d \geq 3$),

(If $d \geq 4$).
Lemma

Let \( \{u_i\}_{i=0}^d \) and \( \{v_i\}_{i=0}^d \) denote bases for \( V \). Then the following are equivalent:

(i) there exists \( A \in \text{End}(V) \) such that \( Au_i = u_{i-1} \) (\( 1 \leq i \leq d \)), \( Au_0 = 0 \), \( Av_i = v_{i-1} \) (\( 1 \leq i \leq d \)), \( Av_0 = 0 \);

(ii) the transition matrix from \( \{u_i\}_{i=0}^d \) to \( \{v_i\}_{i=0}^d \) is upper triangular and Toeplitz.
In Section I we defined an LR pair.

In this section we describe the LR pairs in detail.
Let $A, B$ denote an LR pair on $V$.

By definition, there exists a decomposition $\{V_i\}_{i=0}^d$ of $V$ that is lowered by $A$ and raised by $B$.

Observe that $V_i = B^i V_0$ for $0 \leq i \leq d$. Moreover $V_0 = A^d V$.

So $V_i = B^i A^d V$ for $0 \leq i \leq d$.

Therefore, the decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by $A, B$.

We call $\{V_i\}_{i=0}^d$ the $(A, B)$-decomposition of $V$.

Note that $\{V_{d-i}\}_{i=0}^d$ is the $(B, A)$-decomposition of $V$. 

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Let $A, B$ denote an LR pair on $V$. Then each of $A, B$ is Nil. The $(A, B)$-decomposition of $V$ induces the flag $\{A^{d-i}V\}_{i=0}^d$. The $(B, A)$-decomposition of $V$ induces the flag $\{B^{d-i}V\}_{i=0}^d$. The flags $\{A^{d-i}V\}_{i=0}^d$ and $\{B^{d-i}V\}_{i=0}^d$ are opposite.
The LR pairs on $V$ are described as follows.

Let $\{\varphi_i\}_{i=1}^d$ denote a sequence of nonzero scalars in $\mathbb{F}$.

Let $\{v_i\}_{i=0}^d$ denote a basis for $V$.

Define $A \in \text{End}(V)$ such that $Av_i = v_{i-1}$ for $1 \leq i \leq d$ and $Av_0 = 0$.

Define $B \in \text{End}(V)$ such that $Bv_{i-1} = \varphi_i v_i$ for $1 \leq i \leq d$ and $Bv_d = 0$.

Then the pair $A, B$ is an LR pair on $V$. 
An example of an LR pair, cont.

With respect to the basis \( \{ v_i \}_{i=0}^d \) the matrices representing \( A \) and \( B \) are

\[
A : \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
B : \begin{pmatrix}
0 & \varphi_1 & 0 & \cdots & 0 \\
\varphi_2 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \varphi_d \\
0 & 0 & \cdots & \varphi_d & 0
\end{pmatrix}.
\]
We call the basis \( \{ v_i \}_{i=0}^d \) an \((A, B)\)-basis of \( V \).

Observe that an \((A, B)\)-basis of \( V \) induces the \((A, B)\)-decomposition of \( V \).

We call the sequence \( \{ \varphi_i \}_{i=1}^d \) the parameter sequence of \( A, B \).

For notational convenience define \( \varphi_0 = 0 \) and \( \varphi_{d+1} = 0 \).
The classification of LR pairs up to isomorphism

We now classify the LR pairs up to isomorphism.

Lemma

Consider the map which sends an LR pair to its parameter sequence. This map induces a bijection between the following two sets:

(i) the isomorphism classes of LR pairs over $\mathbb{F}$ that have diameter $d$;
(ii) the sequences $\{\varphi_i\}_{i=1}^d$ of nonzero scalars in $\mathbb{F}$.
We turn our attention to LR triples.

Let $A, B, C$ denote an LR triple on $V$.

Then each of $A, B, C$ is Nil.

Also, the following are mutually opposite flags on $V$:

$$\{ A^{d-i} V \}_{i=0}^d, \quad \{ B^{d-i} V \}_{i=0}^d, \quad \{ C^{d-i} V \}_{i=0}^d.$$
Consider our LR triple $A, B, C$ on $V$. In each row of the table below, we display a decomposition of $V$ along with its induced flag on $V$:

<table>
<thead>
<tr>
<th>decompo. of $V$</th>
<th>induced flag on $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A, B)$</td>
<td>${A^{d-i}V}^{d}_{i=0}$</td>
</tr>
<tr>
<td>$(B, C)$</td>
<td>${B^{d-i}V}^{d}_{i=0}$</td>
</tr>
<tr>
<td>$(C, A)$</td>
<td>${C^{d-i}V}^{d}_{i=0}$</td>
</tr>
<tr>
<td>$(B, A)$</td>
<td>${B^{d-i}V}^{d}_{i=0}$</td>
</tr>
<tr>
<td>$(C, B)$</td>
<td>${C^{d-i}V}^{d}_{i=0}$</td>
</tr>
<tr>
<td>$(A, C)$</td>
<td>${A^{d-i}V}^{d}_{i=0}$</td>
</tr>
</tbody>
</table>
For our LR triple $A, B, C$ on $V$, the associated decompositions of $V$ are related as follows (taking $d = 4$):

```
      B
     / \   \
    /   \  \
   /     \ 
  /       \ 
 /         \ 
```

“A, B, C pull toward their corner”
A notational convention for LR triples

Consider our LR triple $A, B, C$ on $V$.

For any object $f$ that we associate with this LR triple, then $f'$ (resp. $f'''$) will denote the corresponding object for the LR triple $B, C, A$ (resp. $C, A, B$).
Consider our LR triple $A, B, C$ on $V$. By definition $A, B$ (resp. $B, C$) (resp. $C, A$) is an LR pair on $V$. Using our notational convention, for these LR pairs the parameter sequence is denoted as follows:

<table>
<thead>
<tr>
<th>LR pair</th>
<th>parameter sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, B$</td>
<td>${\varphi_i}_{i=1}^d$</td>
</tr>
<tr>
<td>$B, C$</td>
<td>${\varphi_i'}_{i=1}^d$</td>
</tr>
<tr>
<td>$C, A$</td>
<td>${\varphi_i''}_{i=1}^d$</td>
</tr>
</tbody>
</table>

We call the sequence

$$\left(\{\varphi_i\}_{i=1}^d; \{\varphi_i'\}_{i=1}^d; \{\varphi_i''\}_{i=1}^d\right)$$

the **parameter array** of the LR triple $A, B, C$. 

In general, an LR triple is not determined up to isomorphism or equivalence by its parameter array.

In order to describe an LR triple with precision, we bring in additional parameters.
Lemma

Consider our LR triple $A, B, C$ on $V$. Let $\{u_i\}_{i=0}^d$ denote an $(A, C)$-basis of $V$, and let $\{v_i\}_{i=0}^d$ denote an $(A, B)$-basis of $V$. Then the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is upper triangular and Toeplitz.
Definition

Two bases of $\mathcal{V}$ will be called \textbf{compatible} whenever the transition matrix from one basis to the other is upper triangular and Toeplitz, with all diagonal entries 1.
Consider our LR triple $A, B, C$ on $V$.

Let $T \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from a $(C, B)$-basis of $V$ to a compatible $(C, A)$-basis of $V$.

The matrix $T$ is upper triangular and Toeplitz; let $\{\alpha_i\}_{i=0}^d$ denote the corresponding parameters.

Let $\{\beta_i\}_{i=0}^d$ denote the parameters for $T^{-1}$.

We call the 6-tuple

\[
(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha_i'\}_{i=0}^d, \{\beta_i'\}_{i=0}^d; \{\alpha_i''\}_{i=0}^d, \{\beta_i''\}_{i=0}^d)
\]

the **Toeplitz data** for $A, B, C$. By construction $\alpha_0, \alpha'_0, \alpha''_0$ and $\beta_0, \beta'_0, \beta''_0$ are all 1.
Consider our LR triple $A, B, C$ on $V$. With respect to an $(A, B)$-basis of $V$ the matrices representing $A, B, C$ are:

$$A : \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$B : \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \varphi_1 & 0 & \cdots & 0 \\ \varphi_2 & \varphi_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \varphi_d \end{pmatrix},$$

where $\varphi_1, \ldots, \varphi_d$ are parameters.
Representing LR triples by matrices, cont.

\[
C : \begin{pmatrix}
a_0 & \varphi'_1 / \varphi_1 & 0 \\
\varphi''_1 & a_1 & \varphi'_{d-1} / \varphi_2 \\
\varphi''_{d-1} & a_2 & \ddots & \ddots \\
0 & \ddots & \ddots & \varphi'_1 / \varphi_d \\
\varphi''_1 & \ddots & \ddots & a_d
\end{pmatrix},
\]

where

\[
a_i = \alpha''_1 (\varphi'_{d-i+1} - \varphi'_d) = \alpha'_1 (\varphi''_{d-i+1} - \varphi''_d)
\]

for \(0 \leq i \leq d\).
By the previous slide, our LR triple $A, B, C$ is determined up to isomorphism by its parameter array and Toeplitz data.

We now give a more detailed statement.

**Lemma**

Assume $d \geq 1$. Then the LR triple $A, B, C$ is uniquely determined up to isomorphism by its parameter array along with any one of the following six scalars:

$\alpha_1, \alpha'_1, \alpha''_1; \quad \beta_1, \beta'_1, \beta''_1$. 
From our matrix representations and linear algebra we obtain several equations relating the parameter array and Toeplitz data.

These are shown on the next few slides.

**Lemma**

For $1 \leq i \leq d$,

$$a_0 + a_1 + \cdots + a_{d-i} = \beta_1' \varphi_i'' = \beta_1'' \varphi_i'.$$
Lemma

For $1 \leq i \leq d$,

$$\frac{\varphi'_i}{\varphi''_{d-i+1}} = \alpha'_0 \beta'_2 \varphi_{i-1} + \alpha'_1 \beta'_1 \varphi_i + \alpha'_2 \beta'_0 \varphi_{i+1},$$

$$\frac{\varphi''_i}{\varphi'_d_{d-i+1}} = \alpha''_0 \beta''_2 \varphi_{i-1} + \alpha''_1 \beta''_1 \varphi_i + \alpha''_2 \beta''_0 \varphi_{i+1},$$
Some relations, cont.

**Lemma**

For $3 \leq r \leq d + 1$ and $0 \leq i \leq d - r + 1$,

\[
0 = \alpha'_0 \beta'_r \varphi_i + \alpha'_1 \beta'_{r-1} \varphi_{i+1} + \cdots + \alpha'_r \beta'_0 \varphi_{i+r},
\]

\[
0 = \alpha''_0 \beta''_r \varphi_i + \alpha''_1 \beta''_{r-1} \varphi_{i+1} + \cdots + \alpha''_r \beta''_0 \varphi_{i+r}.
\]
Some relations, cont.

Lemma

For $1 \leq i \leq d$ and $0 \leq j \leq d - i$,

\[
\begin{align*}
\alpha'_{i-1} \frac{\varphi_{j+1}}{\varphi''_{d-j}} + \alpha'_i a'_{d-j} + \alpha'_{i+1} \varphi_j &= \alpha'_{i+1} \varphi_{i+j+1}, \\
\alpha''_{i-1} \frac{\varphi''_{j+1}}{\varphi'_{d-j}} + \alpha''_i a''_{d-j} + \alpha''_{i+1} \varphi_j &= \alpha''_{i+1} \varphi_{i+j+1}.
\end{align*}
\]
Before proceeding we mention a special type of LR triple.

**Definition**

The LR triple $A, B, C$ is called **bipartite** whenever each of $a_i, a'_i, a''_i$ is zero for $0 \leq i \leq d$. 
### Lemma

Assume that the LR triple $A, B, C$ is nonbipartite. Then $d \geq 1$. Moreover each of

$$\alpha_1, \alpha'_1, \alpha''_1, \beta_1, \beta'_1, \beta''_1$$

is nonzero.

### Lemma

Assume that the LR triple $A, B, C$ is bipartite. Then $d$ is even. Moreover for $0 \leq i \leq d$, each of

$$\alpha_i, \alpha'_i, \alpha''_i, \beta_i, \beta'_i, \beta''_i$$

is zero if $i$ is odd and nonzero if $i$ is even.
Earlier we gave some equations relating the parameter array and Toeplitz data for the LR triple $A, B, C$.

Before solving the equations, we simplify things using an equivalence transformation.

Since the transformation involves taking square roots,

\[\text{from now on, assume that } \mathbb{F} \text{ is algebraically closed}\]
A reduction involving equivalence

**Lemma**

Assume that the LR triple $A, B, C$ is nonbipartite. After multiplying $A, B, C$ by a (possibly different) nonzero scalar in $\mathbb{F}$,

(i) $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$;
(ii) $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$;
(iii) $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$. 

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A reduction involving equivalence, cont.

Lemma

Assume that the LR triple $A, B, C$ is bipartite. After multiplying $A, B, C$ by a (possibly different) nonzero scalar in $\mathbb{F}$,

(i) $\varphi_{i-1} \varphi_i = \varphi'_{i-1} \varphi'_i = \varphi''_{i-1} \varphi''_i$ for $1 \leq i \leq d$;
(ii) $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$;
(iii) $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$.

As we proceed, all LR triples mentioned are assumed to be reduced as above.
In this section we classify up to equivalence the LR triples over \( \mathbb{F} \).

To avoid trivialities assume \( d \geq 2 \).

First we list six families of solutions.

This is done over the next few slides.
LR triples of $q$-Weyl type

**Example**

$q$-Weyl type

$q \neq 0, \quad q^{2d+2} = 1, \quad q^{2i} \neq 1 \quad (1 \leq i \leq d),$

\[
\varphi_i = 1 - q^{-2i} \quad (1 \leq i \leq d),
\]

\[
\alpha_1 = \frac{q^{r+1/2} + q^{-r-1/2}}{q - q^{-1}} \quad (r \in \mathbb{Z}, 0 \leq r \leq d).
\]
LR triples of $q$-Weyl type, cont.

For an LR triple $A$, $B$, $C$ on $V$ of $q$-Weyl type,

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I,$$
$$\frac{qBC - q^{-1}CB}{q - q^{-1}} = I,$$
$$\frac{qCA - q^{-1}AC}{q - q^{-1}} = I.$$

Here $V$ becomes a module for the “reduced” $U_q(\mathfrak{sl}_2)$ algebra $U_q(\mathfrak{sl}_2)_{\text{red}}$, on which $A$, $B$, $C$ act as the equitable generators.
LR triples of $U_q(\mathfrak{sl}_2)$ type

Example

$U_q(\mathfrak{sl}_2)$ type

$q \neq 0$, $q^{2i} \neq 1$ $(1 \leq i \leq d)$,

$$\varphi_i = \frac{(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})}{q^{d-2i+1}(q - q^{-1})^2}$$

$(1 \leq i \leq d)$,

$\alpha_1 = 1$. 

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LR triples of $U_q(\mathfrak{sl}_2)$ type, cont.

For an LR triple $A, B, C$ on $V$ of $U_q(\mathfrak{sl}_2)$ type,

\begin{align*}
q^{-3}A^2B - (q + q^{-1})ABA + q^3BA^2 &= (q + q^{-1})A, \\
q^{-3}B^2C - (q + q^{-1})BCB + q^3CB^2 &= (q + q^{-1})B, \\
q^{-3}C^2A - (q + q^{-1})CAC + q^3AC^2 &= (q + q^{-1})C
\end{align*}

and also

\begin{align*}
q^{-3}AB^2 - (q + q^{-1})BAB + q^3B^2A &= (q + q^{-1})B, \\
q^{-3}BC^2 - (q + q^{-1})CBC + q^3C^2B &= (q + q^{-1})C, \\
q^{-3}CA^2 - (q + q^{-1})ACA + q^3A^2C &= (q + q^{-1})A.
\end{align*}

Here $V$ becomes a $U_q(\mathfrak{sl}_2)$-module on which $A, B, C$ act as the nilpotent relatives of the equitable generators.
LR triples of $\mathfrak{sl}_2$ type

Example

$\mathfrak{sl}_2$ type

Char($\mathbb{F}$) = 0 or Char($\mathbb{F}$) > $d$,

$\varphi_i = i(i - d - 1)$ \hspace{1cm} (1 \leq i \leq d),

$\alpha_1 = 1.$
For an LR triple $A, B, C$ on $V$ of $\mathfrak{sl}_2$ type,

\[
AB - BA = A + B - C,
\]
\[
BC - CB = B + C - A,
\]
\[
CA - AC = C + A - B.
\]

Here $V$ becomes an $\mathfrak{sl}_2$-module on which the $A, B, C$ act as the nilpotent elements which form the basis for $\mathfrak{sl}_2$ dual to the equitable basis with respect to the Killing form.
LR triples of $U_q(sl_2)_{alt}$ type

Example

$U_q(sl_2)_{alt}$ type

- $d$ is even,
- $q \neq 0$, $q^{d+1} \neq 1$, $q^i \neq 1$ ($1 \leq i \leq d/2$),
- $\varphi_i = \begin{cases} q^{i/2} - 1 & \text{if } i \text{ is even;} \\ q^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases}$ ($1 \leq i \leq d$),
- $\alpha_1 = 1$. 
LR triples of $U_q(\mathfrak{sl}_2)_{alt}$ type, cont.

For an LR triple $A, B, C$ on $V$ of $U_q(\mathfrak{sl}_2)_{alt}$ type,

$$\frac{A^2B - qBA^2}{1 - q} = -A,$$
$$\frac{B^2C - qCB^2}{1 - q} = -B,$$
$$\frac{C^2A - qAC^2}{1 - q} = -C,$$

$$\frac{AB^2 - qB^2A}{1 - q} = -B,$$
$$\frac{BC^2 - qC^2B}{1 - q} = -C,$$
$$\frac{CA^2 - qA^2C}{1 - q} = -A.$$

Here $V$ becomes a module for $U_q(\mathfrak{sl}_2)_{alt}$ on which $A, B, C$ act as the equitable generators.
LR triples of the First Bipartite type

Example

First Bipartite type

\( d \) is even, \( q \neq 0 \), \( \rho \rho' \rho'' = -q^{1-d/2} \),

\( \alpha_1 = 0 \), \( q^i \neq 1 \) \((1 \leq i \leq d/2)\),

for \( 1 \leq i \leq d \),

\[
\varphi_i = \begin{cases} 
\frac{\rho \frac{1-q^i/2}{1-q}}{\rho} & \text{if } i \text{ is even}; \\
\frac{q \frac{1-q(i-d-1)/2}{1-q}}{\rho} & \text{if } i \text{ is odd}; 
\end{cases}
\]

\[
\varphi'_i = \begin{cases} 
\frac{\rho' \frac{1-q^i/2}{1-q}}{\rho'} & \text{if } i \text{ is even}; \\
\frac{q \frac{1-q(i-d-1)/2}{1-q}}{\rho'} & \text{if } i \text{ is odd}; 
\end{cases}
\]

\[
\varphi''_i = \begin{cases} 
\frac{\rho'' \frac{1-q^i/2}{1-q}}{\rho''} & \text{if } i \text{ is even}; \\
\frac{q \frac{1-q(i-d-1)/2}{1-q}}{\rho''} & \text{if } i \text{ is odd}. 
\end{cases}
\]
For an LR triple $A, B, C$ on $V$ of the First Bipartite type,

\[
A^3B + A^2BA - qABA^2 - qBA^3 = (\rho + q/\rho)A^2, \\
B^3C + B^2CB - qBCB^2 - qCB^3 = (\rho' + q/\rho')B^2, \\
C^3A + C^2AC - qCAC^2 - qAC^3 = (\rho'' + q/\rho'')C^2
\]

and also

\[
AB^3 + BAB^2 - qB^2AB - qB^3A = (\rho + q/\rho)B^2, \\
BC^3 + CBC^2 - qC^2BC - qC^3B = (\rho' + q/\rho')C^2, \\
CA^3 + ACA^2 - qA^2CA - qA^3C = (\rho'' + q/\rho'')A^2.
\]

Here $V$ becomes a $U_q(\mathfrak{sl}_2)$-module on which $A^2, B^2, C^2$ act as the nilpotent relatives of the equitable generators.
LR triples of the Second Bipartite type

**Example**

Second Bipartite type

- \(d\) is even, \(\text{Char}(\mathbb{F}) = 0\) or \(\text{Char}(\mathbb{F}) > d/2\),

\[ \rho \rho' \rho'' = -1, \quad \alpha_1 = 0, \]

for \(1 \leq i \leq d\),

\[
\varphi_i = \begin{cases} 
\frac{i \rho}{2} & \text{if } i \text{ is even;} \\
\frac{i-d-1}{2 \rho} & \text{if } i \text{ is odd;}
\end{cases}
\]

\[
\varphi'_i = \begin{cases} 
\frac{i \rho'}{2} & \text{if } i \text{ is even;} \\
\frac{i-d-1}{2 \rho'} & \text{if } i \text{ is odd;}
\end{cases}
\]

\[
\varphi''_i = \begin{cases} 
\frac{i \rho''}{2} & \text{if } i \text{ is even;} \\
\frac{i-d-1}{2 \rho''} & \text{if } i \text{ is odd.}
\end{cases}
\]
LR triples of the Second Bipartite type, cont.

For an LR triple $A, B, C$ on $V$ of the Second Bipartite type,

\[
A^3 B + A^2 B A - A B A^2 - B A^3 = (\rho + 1/\rho) A^2,
\]
\[
B^3 C + B^2 C B - B C B^2 - C B^3 = (\rho' + 1/\rho') B^2,
\]
\[
C^3 A + C^2 A C - C A C^2 - A C^3 = (\rho'' + 1/\rho'') C^2
\]

and also

\[
A B^3 + B A B^2 - B^2 A B - B^3 A = (\rho + 1/\rho) B^2,
\]
\[
B C^3 + C B C^2 - C^2 B C - C^3 B = (\rho' + 1/\rho') C^2,
\]
\[
C A^3 + A C A^2 - A^2 C A - A^3 C = (\rho'' + 1/\rho'') A^2.
\]

Here $V$ becomes an $\mathfrak{sl}_2$-module on which $A^2, B^2, C^2$ act as the nilpotent elements which form the basis for $\mathfrak{sl}_2$ dual to the equitable basis with respect to the Killing form.
The main theorem

Assume that \( F \) is algebraically closed. Then each LR triple over \( F \) of diameter \( d \geq 2 \) is equivalent to exactly one of the LR triples listed in the above six families.
In this talk, we introduced the notion of an LR triple of linear transformations.

We classified up to equivalence the LR triples over an algebraically closed field.

The solutions fell into six infinite families.

We showed how each family is linked to $U_q(sl_2)$ or a related algebra.

Thank you for your attention!

THE END