Lecture 33 Wed Nov 28

We will do Ch 14 next, and return to Ch 9 time permitting

Ch 14 Polya counting

14.1 Permutations and symmetry groups

Let $X = \text{nonempty finite set}$

Say $X = \{1, 2, \ldots, n\}$

Consider perm of $X$:

$a_1, a_2, \ldots, a_n$

View this as a bijection

$X \rightarrow X$

$i \rightarrow a_i$
To emphasize this view we often write

\[
\begin{pmatrix}
  1 & 2 & \cdots & n \\
  a_1 & a_2 & \cdots & a_n
\end{pmatrix}
\]

the bijection sends each number in top row to the number beneath it.

Ex \quad n = 5

the permutation

\[
\begin{pmatrix}
  1 & 2 & 3 & 4 & 5 \\
  4 & 3 & 5 & 1 & 2
\end{pmatrix}
\]

satisfy

\[
\begin{pmatrix}
  1 & 1 \\
  2 & 2 \\
  3 & 3 \\
  4 & 4 \\
  5 & 5
\end{pmatrix}
\]
Def: For $n \geq 1$

$$S_n = \text{set of all perms of } \{1, 2, \ldots, n\}$$

Composition of permutations

Given perms $f : X \to X$, $g : X \to X$

their composition $f \circ g : X \to X$ satisfies

$$(f \circ g)(x) = f(g(x)) \quad \forall x \in X$$

"First apply $g$ and then apply $f$"

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{f}
\end{array}
\]

$$(f \circ g)(x) = f(g(x))$$
Ex \( f \) \( \text{from pems} \)

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} \]

Find \( fog \)

Sol

\[ \begin{array}{cccc}
1 & & & 1 \\
2 & 2 & 2 & \\
3 & 3 & & \\
4 & 4 & 4 & \\
5 & 5 & & \\
\end{array} \]

\[\Rightarrow \quad g \quad \Rightarrow \quad f\]

\[fog = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \]

\[(fog)(1) = f(g(1)) = f(2) = 3 \quad \text{etc.}\]
Ex. Referring to above fig. find $gof$

\[ \text{Graph of } f \] \hspace{2cm} \text{Graph of } g \\
\]

\[ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix} \]

Note: $gof \neq fog$ in general.

We view composition $\circ$ as a binary operation on $S_n$ given $f$ and $g$ in $S_n$, their composition $fog$ is an element of $S_n$. 
LEM For \( f, g, h \in S_n \)

\[(f \circ g) \circ h = f \circ (g \circ h) \]

"Composition is associative"

pf Each side of \( \star \) is a function \( X \rightarrow X \)

where \( X = \{1, 2, \ldots, n\} \)

Show each function sends each \( x \in X \) to the same thing

\[(f \circ g) \circ h : x \rightarrow (h \circ g)(x) = (f \circ g)(h(x)) = f(g(h(x)))\]

\[f \circ (g \circ h) : x \rightarrow (g \circ h)(x) = g(h(x)) \rightarrow f(g(h(x)))\]

\[f^2 = f \circ f, \quad f^3 = f \circ f \circ f, \quad f' = f\]

From now on we drop parenthesis and write

\[f \circ g \circ h\]

We abbreviate

\[f^2 = f \circ f, \quad f^3 = f \circ f \circ f, \quad f' = f\]

etc
The identity permutation

\[ I : X \rightarrow X \]

satisfies

\[ I(x) = x \quad \forall x \in X \]

"I leaves everything alone"

So

\[ I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \]
LEM \quad F_n \text{ a perm} \quad f : X \to X,

f \circ I = f

I \circ f = f

pf \quad \forall x \in X

(f \circ I)(x) = f \left( \frac{I(x)}{x} \right)

= f(x)

Also

(I \circ f)(x) = I \left( \frac{f(x)}{x} \right)

= f(x)

\square
Inverses

Given perm \( f: X \rightarrow X \)

Say \( f = (\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
4 & 3 & 1 & 5 & 2
\end{array}) \)

\[ f^{-1} = \begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
3 & 5 & 2 & 1 & 4
\end{array} \]

Call \( f^{-1} \) the inverse of \( f \)

\( f^{-1} \) undoes \( f \)
Given a perm $f: X \to X$

then $f^{-1}$ is described as follows:

**View I**

For all $x, y \in X$

$f(x) = y$ if and only if $f^{-1}(y) = x$

**View II**

For all $x \in X$

$f^{-1}(f(x)) = x$

**View III**

For all $y \in X$

$f(f^{-1}(y)) = y$

$$f^{-1} \circ f = I$$

$$f \circ f^{-1} = I$$
LEMMA Given perm \( f: X \to X \nolimits \)

then for all perms \( g: X \to X \nolimits \)

the following are equivalent:

(i) \( \; g \circ f = I \)

(ii) \( \; f \circ g = I \)

(iii) \( \; g = f^{-1} \)

pf \( (iii) \to (i) \quad \text{From View II alone} \)

\( (iv) \to (i) \quad \text{From View III alone} \)

(i) \to (iii)

\[ g \circ f = I \]

\[ (g \circ f) \circ f^{-1} = I \circ f^{-1} \]

\[ f^{-1} \]

\[ g \circ (f \circ f^{-1}) \]

\[ g \circ I \]

\( (ii) \to (iii) \quad \text{Similar} \)
Ex. Given \( f : X \rightarrow X \), \( g : X \rightarrow X \)

Find \((f \circ g)^{-1}\)

Sol

\[ f \circ g : X \rightarrow X \rightarrow X \]

\(f \circ g\) is undone by

\[(f \circ g)^{-1} : X \leftarrow X \leftarrow X \]

So

\[(f \circ g)^{-1} = g^{-1} \circ f^{-1}\]

check

\[(g^{-1} \circ f^{-1}) \circ (f \circ g) = I\]

\[g^{-1} \circ f^{-1} \circ g \circ f = I\]

\[g^{-1} \circ f = T\]
More generally, given permutations

\[ f_0 : X \to X, \]
\[ f_1 : X \to X, \]
\[ \ldots \]
\[ f_r : X \to X \]

\[ (f_0 \circ f_1 \circ \cdots \circ f_r)^{-1} = f_r^{-1} \circ f_{r-1}^{-1} \circ \cdots \circ f_1^{-1} \circ f_0^{-1} \]

In particular, for any permutation \( f : X \to X \)

and \( r \geq 1 \)

\[ (f^r)^{-1} = (f^{-r})^r \]

Call this commutator \( f^{-r} \)

Formally define

\[ f^0 = I \]

By construction

\[ f^r \circ f^s = f^{r+s} \quad \text{for all integers } r, s \]

\[ (f^r)^s = f^{rs} \]
14.1 Cont.

\[ X = \text{nonempty finite set} \]

Say
\[ X = \{1, 2, \ldots, n\} \]

Given perm
\[ f : X \to X \]

Consider
\[ I, f, f^2, f^3, \ldots \]

Finitely many perms \( X \to X \)

Must be duplication among \( X \) s
\[ f^r = f^s \quad r < s \]

So
\[ f^{s-r} = I \]

So
\[ \exists \ m \geq 1 \text{ such that} \]
\[ f^m = I \]

Note
\[ f^{-1} = f^{m-r} \]
f \circ f^{m-1} = f^m = I

DEF A permutation group on $X$ is a set $G$ of permutations $X \to X$ such that:

1. For all $f, g \in G$
   
   $f \circ g \in G$

2. $I \in G$

G is closed under composition.
LEM Given a permutation \( \sigma \) on a set \( X \).

Then for all \( f \in G \)

\[ f^{-1} \in G \]

pf

Case \( f = I \): ok since \( I^{-1} = I \)

Case \( f \neq I \): \( \exists m \geq 1 \) such that \( f^m = I \)

\[ f^{-1} = f^{m-1} \]

\[ = \underbrace{f \circ f \circ \ldots \circ f}_{m-1} \]

\( \in G \) since \( G \) is closed under comp.
Examples of Permutation Groups

EX 1  Recall

\[ S_n = \text{set of all perms of } X = \{1, 2, \ldots, n\} \]

\[ S_n = \text{perm group on } X \]

This group is called the symmetric group of order \( n \).
Ex 2 Consider an oriented regular $n$-gon $P$ in the plane.

Ex $n = 6$

View $X =$ set of corners (vertices) of $P$

$X = \{ 1, 2, 3, 4, 5, 6 \}$
$P$ has rotational symmetries:

If we rotate $P$ clockwise by some multiple of $60^\circ$ ($0 \leq m \leq 5$),

then result coincides with $P_0$.

Each rotation induces perm of $X$:

<table>
<thead>
<tr>
<th>$m$</th>
<th>perm of $X$</th>
<th>name of perm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \end{pmatrix}$</td>
<td>$I$</td>
</tr>
<tr>
<td>1</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 &amp; 1 \end{pmatrix}$</td>
<td>$R$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 3 &amp; 4 &amp; 5 &amp; 6 &amp; 1 &amp; 2 \end{pmatrix}$</td>
<td>$R^2 = R_0 R$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 4 &amp; 5 &amp; 6 &amp; 1 &amp; 2 &amp; 3 \end{pmatrix}$</td>
<td>$R^3$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 5 &amp; 6 &amp; 1 &amp; 2 &amp; 3 &amp; 4 \end{pmatrix}$</td>
<td>$R^4$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 &amp; 6 \ 6 &amp; 1 &amp; 2 &amp; 3 &amp; 4 &amp; 5 \end{pmatrix}$</td>
<td>$R^5$</td>
</tr>
</tbody>
</table>

Note $R^6 = I$.
Define

\[ G = \text{net of all rotational symmetries of } P \]
\[ = \{ I, R, R^2, R^3, R^4, R^5 \} \]

So \[ |G| = 6 \]

Then \[ G \] is a perm group \( \times \)

"cyclic group of order 6"

Inverse:

<table>
<thead>
<tr>
<th>( f )</th>
<th>( I )</th>
<th>( R )</th>
<th>( R^2 )</th>
<th>( R^3 )</th>
<th>( R^4 )</th>
<th>( R^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^{-1} )</td>
<td>( I )</td>
<td>( R^5 )</td>
<td>( R^4 )</td>
<td>( R^3 )</td>
<td>( R^2 )</td>
<td>( R )</td>
</tr>
</tbody>
</table>

In general, \( f_{n=2} \)

the cyclic group of order \( n \) = group of rotational symmetries of an oriented regular \( n \)-gon
Ex 3  Consider a non-oriented regular n-gon in the plane

\[ n = 6 \]

View \( X = \text{set of vertices of } P \) as before

\[ X = \{1, 2, 3, 4, 5, 6\} \]
P has rotational and reflective symmetries

Rotational symmetries: 6 of these form a regular hexagon

Reflective symmetries

If we reflect P across a line of reflective symmetry, the result coincides with P.

ex
$P$ has 6 lines of reflective symmetry.

Each reflective symmetry of $P$ induces perm of $X$. 
<table>
<thead>
<tr>
<th>line of sym</th>
<th>perm of X</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (1 \ 6 \ 5 \ 4 \ 3 \ 2) )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (2 \ 1 \ 6 \ 5 \ 4 \ 3) )</td>
<td>( R \circ \gamma )</td>
</tr>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (3 \ 2 \ 1 \ 6 \ 5 \ 4) )</td>
<td>( R^2 \circ \gamma )</td>
</tr>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (4 \ 3 \ 2 \ 1 \ 6 \ 5) )</td>
<td>( R^3 \circ \gamma )</td>
</tr>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (5 \ 4 \ 3 \ 2 \ 1 \ 6) )</td>
<td>( R^4 \circ \gamma )</td>
</tr>
<tr>
<td>( \frac{6}{1} \frac{5}{2} \frac{4}{3} )</td>
<td>( (1 \ 2 \ 3 \ 4 \ 5 \ 6) ) ( (6 \ 5 \ 4 \ 3 \ 2 \ 1) )</td>
<td>( R^5 \circ \gamma )</td>
</tr>
</tbody>
</table>
Recall $R^6 = I$

For any reflection
\[
(\text{reflection})^2 = I
\]

so
\[
\text{each reflection is its own inverse}
\]

obs
\[
\gamma^2 = I
\]

For $0 \leq \gamma \leq 5$

\[
(R^{\delta} \gamma)^2 = I
\]

\[
R^{\delta} \gamma = (R^{\delta} \gamma)^{-1}
\]

\[
= R^{\delta} \gamma^{-1}
\]

\[
= \gamma R^{\delta - \gamma}
\]

\[
= \gamma R^{\delta - \gamma}
\]

let $G =$ set of all symmetries of $P$, both rotational and reflectonal

\[
G = \{ R^{\delta} \gamma \mid \delta \in \{0, 5\}, \gamma \in \{0, 1, 2, 3, 4, 5\} \}
\]

$|G| = 12$

then $G$ is a perm group in $X$

"dihedral group of order 12"
In general for \( n \geq 3 \)

The **dihedral group of order** 2\(n \) = group of symmetries of the regular \( n \)-gon

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Any geometric figure of any dimension has a symmetry group

**Ex.** The 5 platonic solids in 3 dimensions

- cube
- tetrahedron
- octahedron
- dodecahedron
- icosahedron

---

**Cube**

![Diagram of a cube]

\( X = \text{set of vertices} \quad |X| = 8 \)

Each symmetry of the cube induces perm of \( X \)

Let \( G = \text{set of resulting perms of } X \)

\( G = \text{perm group on } X \)
Claim \( |G| = 48 \)

Proof: Consider 3 vertices

![Diagram of a cube with vertices labeled a, b, c.]

To construct \( f \in G \) we define \( f(a), f(b), f(c) \) in stages.

<table>
<thead>
<tr>
<th>Stage</th>
<th>To Do</th>
<th>#Choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pick ( f(a) )</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>Pick ( f(b) ) from among the 3 vertices adjacent to ( a )</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Pick ( f(c) ) from among 2 vertices adjacent ( f(b) ) other than ( f(a) )</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
\text{#poss} = 8 \times 3 \times 2 = 48
\]

\( \square \)
Let $X$ = nonempty finite set

Say $X = \{1, 2, \ldots, n\}$

Let $G$ = permutation group on $X$

Running example (REX): $X$ is set of vertices for the regular 4-gm

```
 1 2
 4 3
```

$G =$ the group of symmetries

= dihedral group of order 8

Define

$\rho$: 90$^\circ$ clockwise rotation

$\tau$: reflection about $\Box$

$G = \{\text{I, } \rho, \rho^2, \rho^3, \tau, \rho \tau, \rho^2 \tau, \rho^3 \tau\}$
**Def:** A *coloring* of $X$ is an assignment of a color to each element of $X$ (distinct elements of $X$ might get the same color).

**REX:** Using colors Red ($R$) and Blue ($B$) there are $2^4 = 16$ possible colorings. They are:

<table>
<thead>
<tr>
<th>#B</th>
<th>desc</th>
<th>#colorings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$R\ R\ R\ R$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$B\ R\ R\ R$ + cyclic perms</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>$B\ B\ R\ R$</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>$B\ R\ B\ B$</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$B\ B\ B\ B$</td>
<td>4</td>
</tr>
</tbody>
</table>

*Table 1*
View each coloring above as a function

\[ X \longrightarrow \{ \text{R, B}\} \]

\[ i \longrightarrow \text{color assigned to } i \]

Given \( f \in G \)

Given a coloring \( c \) of \( X \)

Permute \( X \) to get another coloring of \( X \):

\[ X \xrightarrow{c} \{ \text{R, B}\} \]

\[ f \downarrow \]

\[ X \]

\[ X \quad \rightarrow \quad \{ \text{R, B}\} \]

\[ f^{-1} \uparrow \]

\[ X \]

Composition \( f \circ c \) is a coloring of \( X \)
Define
\[ f \ast c = c \circ f^{-1} \]

Thus
\[ f \ast c \text{ is a coloring of } X \]

that assigns each element \( x \in X \) the color \( c(f^{-1}(x)) \)

Note \( f \) induces a perm of the set of all colorings \( \mathcal{C}(X) \):

\[
\begin{align*}
\{ \text{coloring of } X \} & \rightarrow \{ \text{coloring of } X \} \\
c & \rightarrow f \ast c
\end{align*}
\]

LEM - For a coloring \( c \) of \( X \):
\[ I \ast c = c \]

\[ I = \text{identity map of } X \]

pf
\[
\begin{align*}
I \ast c &= c \circ I^{-1} \\
&= c \circ I \\
&= c
\end{align*}
\]
LEM Given $f, g \in G$,

Given a coloring $c_X$

Then

\[ f^* (g^* c) = (f o g)^* c \]

pf

\[ f^* (g^* c) = (g^* c) o f^* \]
\[ = (c o g)^* o f^* \]
\[ = c o (g^* o f^*) \]
\[ = c o (f o g)^* \]
\[ = (f o g)^* c \]

\[ \square \]

LEM Given $f \in G$,

Given colorings $c_1$ and $c_2 \in X$ such that

\[ c_2 = f^* c_1 \]

Then

\[ c_1 = f^{-1}^* c_2 \]

pf

\[ f^{-1}^* c_2 = f^{-1}^* (f^* c_1) \]
\[ = (f^{-1} o f)^* c_1 \]
\[ = I^* c_1 \]
\[ = c_1 \]

\[ \square \]
For the coloring \( c \):

\[
\begin{array}{c}
R & R \\
R & R \\
\end{array}
\]

\[ f \ast c = c \quad \text{forall } f \in G \]

(ii) Find \( f \ast c \)

\[
\begin{array}{c}
B & R \\
R & R \\
\end{array}
\]

\[ \rho = \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
\end{array} \right) \]

1 2
4 3

\[ \rho \downarrow \uparrow \rho^{-1} \]

1 2
4 3

\[ \rho \ast c : \]

\[
\begin{array}{c}
R & R \\
R & B \\
\end{array}
\]

"Apply \( \rho \) to picture *"
Find $r \ast c$

$$r = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

1 2  \rightarrow  \quad R \ast B

4 3

\[ r \ast c \quad \text{applying } r \text{ to picture } c \]
We record a principle

Given a coloring $c$ of $X$, say

\[ \begin{array}{cc}
B & B \\
R & R \\
\end{array} \]

Then for all $f \in G$, the coloring $f \circ c$ is obtained by applying $f$ to the picture.
Consider the set of colorings of $X$ that have one $B$

$G$ acts on this set as follows

![Diagram]

**Key:** For $f \in G$ and a coloring $c$ of $X$,

$$f \rightarrow c^{f \cdot c}$$
Consider the set of colorings of $X$. Let's have two $B$'s. $G$ acts on this set as follows: 

- $r$: 
  - $B$ becomes $R$ and $R$ becomes $B$.
  - $B$ remains $B$.

- $p$: 
  - $R$ becomes $B$ and $B$ becomes $R$.
  - $B$ remains $B$.

- $t$: 
  - $R$ becomes $B$ and $B$ becomes $R$.
  - $B$ remains $B$.
Above diagram has 2 connected components.

Given colonies $c_1, c_2$ of $X$

call them $G$-equivalent and write $c_1 \sim c_2$

whereas they are in the same connected component of the diagram,

then

$$G
\begin{array}{c}
\sim
c_1 \sim c_2
\end{array}$$

means

there exists $f \in G$ such that

$$c_2 = f \circ c_1$$

The relation $\sim$ is an equivalence relation and the equivalence classes are the connected components of the diagram.
Formal verification that $G$ is an equivalence relation

0. For all colorings $c$ of $X$ show

$$G$$
$c \sim c$

Proof: $c = I \ast c$
$I = \text{identity of } G$

0. For all colorings $c_1, c_2$ of $X$ such that $G$

$$G$$
$c_2 \sim c_1$

Proof: Since $c_1 \sim c_2$ for $f \in G$ s.t. $c_2 = f \ast c_1$

Now $c_1 = f^{-1} \ast c_2$ and $f^{-1} \in G$ so

$$G$$
$c_2 \sim c_1$

1. For all colorings $c_1, c_2, c_3$ of $X$ such that

$$G$$
$c_1 \sim c_2$ and $c_2 \sim c_3$

Show $c_1 \sim c_3$
pf \ \exists f \in G \ st.
\quad c_2 = f \ast c_1

\forall g \in G \ s.t.
\quad c_3 = g \ast c_2

\therefore
\quad c_3 = g \ast c_2
\quad = g \ast (f \ast c_1)
\quad = (g \circ f) \ast c_1
\quad \underbrace{\text{in}}_{G}
\quad G

\therefore
\quad c_1 \sim c_3
\quad \blacksquare
Consider the set of the coloring of \( X \) using colors \( R, B \).

The relation \( \mathcal{E} \) on this set has 6 equivalence classes, one for each row of Table 1.

So up to \( G \)-equivalence, there are 6 colorings of \( X \) with \( R, B \).
14.2 Burnside's Theorem

Let

\[ X = \text{nonempty finite set} \]
\[ X = \{1, 2, \ldots, n\} \]

\[ G = \text{a permutation group on } X \]

Pick some colors and consider the set of all colorings of \( X \) with those colors.

We saw that \( G \) induces a perm gp on that set.

Let \( C \) be a nonempty subset of the above set of colorings of \( X \).

Call \( C \) \underline{G-closed} whenever

\[ f(x) \in C \text{ for all } f \in G \text{ and } c \in C \]

In this case \( G \) induces perm gp on \( C \).

REX For each row in table I, the given set of colorings is \( G \)-closed.
From now on assume $C$ is $G$-closed.

Recall that for colorings $c_1, c_2 \in C$

$c_1 \sim c_2$ means "$G$-equivalent"

If $f \in G$ s.t. $f \circ c = c_2$

$\sim$ is an equivalence on $C$

$C = \text{disjoint union of } \sim$ equivalence classes

Define

$N(c, C) = \# \text{ equivalence classes of } \sim \text{ on } C$

$= \# \text{ of mutually } G\text{-inequivalent colors in } C$

REX If we take

$C = \text{all } 16 \text{ colorings } t \times$

then $N(c, C) = 6$

Since there is 1 $\sim$ equiv. class for each of the 6 rows of table I.

next goal: find formula for $N(c, C)$
For \( c \in C \) and \( f \in G \), we say

\[
\text{f stabilizes } c
\]

or

\[
\text{f fixes } c
\]

where

\[
f \ast c = c
\]

For \( c \in C \) define

\[
G(c) = \{ f \in G / f \ast c = c \}
\]

"The stabilizer of \( c \) in \( G \)"
LEM  Given \( c \in C \), then

\[ G(c) \] is a permutation on \( C \)

pf  Given \( f, g \in G(c) \), show \( fog \in G(c) \):

We have \( f \circ c = c \), \( g \circ c = c \)

\[ (f \circ g) \circ c = f \circ (g \circ c) \]
\[ = f \circ c \]
\[ = c \]

Show \( I \in G(c) \):

\[ I \circ c = c \]

\[ \square \]
LEM. Given \( c \in C \).

Then for all \( f, g \in \mathcal{F} \), the following are equivalent:

(i) \( f \ast c = g \ast c \)

(ii) \( f \circ g \in G(c) \)

\[ \Rightarrow \]

\[ f^{-1} \ast (f \ast c) = f^{-1} \ast (g \ast c) \]

\[ \Rightarrow \]

\[ (f \circ g) \ast c = f^{-1} \circ g \ast c \]

\[ \Rightarrow \]

\[ f^{-1} \circ g \in G(c) \]
Given \( c \in C \)

Consider the equivalence class \( \sim \) containing \( c \)

This is

\[ \{ f \cdot c / f \in G \} \]

Then

For \( c \in C \)

\[ | \{ f \cdot c / f \in G \} | = \frac{|G|}{|G(c)|} \]

pf

Abb \( y = \{ f \cdot c / f \in G \} \)

For \( y \in Y \) define

\[ G^{(y)} = \{ f \in G / f \cdot c = y \} \]

\[ \exists G^{(y)} \exists y \in Y \quad \text{par \, hint: } G \]

So

\[ |G| = \sum_{y \in Y} |G^{(y)}| \]

For \( y \in Y \) show

\[ |G^{(y)}| = |G(c)| \]
For $f \in G^{(y)}$, the map

$$G(c) \rightarrow G^{(y)}$$

$h \rightarrow foh$

is a bijection (by prev LEM)

So

$$|G(c)| = |G^{(y)}|$$

Now

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

and

$$|G(c)|$$

$$= |Y| |G(c)|$$

So

$$|Y| = \frac{|G|}{|G(c)|}$$
For \( f \in G \) define

\[
C(f) = \{ c \in C \mid f \ast c = c \}
\]

"set of colorings in \( C \) that are fixed by \( f \)"

So for \( f \in G \) and \( c \in C \)

\[
c \in C(f) \iff f \ast c = c \iff f \in G(c)
\]

**Thm. (Burnside)**

\[
N = \frac{\sum_{f \in G} |C(f)|}{|G|}
\]

\# of equivalence classes

**pf**

Let

\[
S = \text{set of ordered pairs } (f,c) \text{ such that } f \in G \text{ and } c \in C \text{ and } f \ast c = c
\]

We compute \(|S|\) in two ways

I  \[
|S| = \sum_{f \in G} |C(f)|
\]

II  \[
|S| = \sum_{c \in C} |G(c)|
\]
Let $C_1, C_2, \ldots, C_n$ denote the equivalence classes of $\sim$ on $C$.

\[
|S| = \sum_{c \in C} |G(c)|
\]

\[
= \sum_{i=1}^{n} \sum_{c \in C_i} |G(c)| \quad \text{by previous}
\]

\[
= \sum_{i=1}^{n} |C_i| \cdot \frac{|G|}{|C_i|}
\]

\[
= \sum_{i=1}^{n} |G| = |G|
\]

So

\[
N = \frac{|S|}{|G|}
\]

\[
= \sum_{f \in G} \frac{|C(f)|}{|G|}
\]

\[
\square
\]
\[ C = \text{set of all 16 colorings of } X \text{ using colors } R, B \]

\[ G = \text{dihedral gp of order 8} \]

We saw \( N(G, c) = 6 \)

Let us verify this using Burnside.

By Burnside

\[ N(G, c) = \frac{\sum_{g \in G} |C(g)|}{|G|} \]

\[ |G| = 8 \]

\[ G = \{ 1, \rho, \rho^2, \rho^3, \tau, \rho \tau, \rho^2 \tau, \rho^3 \tau \} \]

For all \( g \in G \) find \(|C(g)|\)
| \( f \in G \) | \( C(f) \) | \( \text{desc} \) | \( |C(f)| \) |
|---|---|---|---|
| 1 | 1 | \( a \) | 2 \( ^4 = 16 \) |
| 1 | 1 \( \rho \) | \( a \) \( \rho \) | 2 \( ^2 = 4 \) |
| 1 | 1 \( \rho ^2 \) | \( a \) \( \rho ^2 \) | 2 \( ^2 = 4 \) |
| 1 | 1 \( \rho ^3 \) | \( a \) \( \rho ^3 \) | 2 \( ^2 = 4 \) |
| 1 | 1 \( \rho \tau \) | \( a \) \( \rho \tau \) | 2 \( ^3 = 8 \) |
| 1 | 1 \( \rho ^2 \tau \) | \( a \) \( \rho ^2 \tau \) | 2 \( ^3 = 8 \) |
| 1 | 1 \( \rho ^3 \tau \) | \( a \) \( \rho ^3 \tau \) | 2 \( ^3 = 8 \) |

\[ N(G, C) = \frac{48}{8} = 6 \]
Ex

\[ X, G \text{ as above} \]

Given integer \( p \geq 1 \)

Given \( p \) distinct colors

Let \( C = \text{set of colorings of } X \) with these colors.

So \[ |C| = p^4 \]

Find \[ N(G, C) \]

Sol

prev ex n case \( p = 2 \)

In prev ex replace \( 2 \) by \( p \)

| \( f \) | \( |C(f)| \) |
|------|--------|
| 1    | 1^4    |
| \( p \) | \( p \) |
| \( p^2 \) | \( p^2 \) |
| \( p^3 \) | \( p^3 \) |
| \( p^4 \) | \( p^4 \) |
| \( p^5 \) | \( p^5 \) |
| \( p^6 \) | \( p^6 \) |

\[
N(G, C) = \frac{p^4 + 2p^3 + 3p^2 + 2p}{8}
\]

<table>
<thead>
<tr>
<th>( p )</th>
<th>( N(G, C) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>55</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Recall Burnside

Given:

- $X$ a nonempty finite set
- $G$ a permutation group on $X$
- $C$ a nonempty set of colors of $X$

Then

$$
N(g, c) = \prod_{\pi \in G} \frac{|C(\pi)|}{|G|}
$$

$\#$ $\mathcal{E}$-equivalence classes in $C$
Ex Consider multiset

\[ S = \{ a, b, c, d \} \]

Fix integer \( n \geq 1 \).

Recall \# n-perms of \( S \) is \( 4^n \).

How many n-perms of \( S \) if we declare each
\[ a, a_1, a_2, \ldots, a_n \]
and its mirror image \( a_n, a_{n-1}, \ldots, a_1 \)
to be equivalent?

[so \( \text{perm } abcd \) equiv dacbba]

So define

\[ X = \{1, 2, \ldots, n\} \]

View \( a, b, c, d \) as colors

\[ C = \text{set of all colorings of } X \text{ with } a, b, c, d \]

\[ |C| = 4^n \]

Define

\[ G = \{ I, \pi \} \]

\[ \pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} \]

\[ \pi^2 = I, \quad G \text{ is a perm grp on } X \]
We seek \( N(G,C) \)

| \( f \in G \) | \( C(f) \) | \( |C(f)| \) |
|-----------|--------|--------|
| \( I \)   |        |        |
| \( T \)   |        |        |

Case \( n = 2r \) is even

\[ a_1, a_2 \ldots \ a_r \mid a_r \ldots a_2, a_1, \ a_i, \ldots, a_r \in \{a, b, c, d\} \]

\( |C(f)| = 4^r \)

Case \( n = 2m + 1 \) is odd

\[ a_1, a_2 \ldots \ a_r \mid a_r \ldots a_2, a_1, \ a_{2n}, \ldots, a_1 \in \{a, b, c, d\} \]

\( |C(f)| = 4^{m+1} \)

\( N(G,C) = \begin{cases} 
\frac{4^n + 4 \sqrt{2}}{2} & \text{if } n \text{ even} \\
\frac{4^n + 4 \frac{\alpha^n}{2}}{2} & \text{if } n \text{ odd}
\end{cases} \)

\( = \frac{4^n + 4 \frac{\alpha^n}{2}}{2} \)
Ex. Tetrahedron

"in 3 dimensions"

"top view"

\[ X = \text{net of vertices} \]

Label \( X \)

The group \( G \) of symmetries of the tetrahedron
<table>
<thead>
<tr>
<th>Types of symmetries on $G$</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>120° clockwise rot</td>
<td>4</td>
</tr>
<tr>
<td>120° c.c. rot</td>
<td>4</td>
</tr>
<tr>
<td>180° rot</td>
<td>3</td>
</tr>
<tr>
<td>identity $I$</td>
<td>1</td>
</tr>
</tbody>
</table>

We allow only "physically possible" symmetries

$|G| = 12$

$G$ is often called the alternating group $A_4$
For the above tetrahedron

Let

\[ C = \text{set of colorings of } \Gamma \text{ with colors } R, B \]

So \( |C| = 2^4 = 16 \)

Find \( N(e, C) \)

**Solution 1:** describe each equivalence class

<table>
<thead>
<tr>
<th># ( \theta )</th>
<th>desc</th>
<th># colorings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image" alt="Triangle" /></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td><img src="image" alt="Triangle" /></td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td><img src="image" alt="Triangle" /></td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td><img src="image" alt="Triangle" /></td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td><img src="image" alt="Triangle" /></td>
<td>1</td>
</tr>
</tbody>
</table>
Each row gives a $\sim$ equivalent

$N(\theta_c) = \#rows = 5$

Sol 2: Use Burnside

| $f \in G$          | $C(f)$ desc | $|C(f)|$ |
|--------------------|-------------|---------|
| $120^\circ$ clockwise rot | $\{a, b\} \subseteq R, R$ | $2^2 = 4$ |
| $120^\circ$ cc. rot | $\ldots$ | $\ldots$ |
| $180^\circ$ rot | $\{a, b\} \subseteq R, R$ | $2^2 = 4$ |
| $I$ | $C$ | $2^4 = 16$ |

$N(\theta_c) = \frac{\sum_{f \in G} |C(f)|}{|G|}$

$= \frac{4 \times 4 + 4 \times 4 + 3 \times 4 + 1 \times 16}{12}$

$= \frac{60}{12}$

$= 5$
Ex. For the above k-configuration

Let
\[ C = \text{set of colorings of } X \text{ with } R, W, B \]

So
\[ |C| = 3^4 = 81 \]

Find \( N(G, C) \)

Sol 1. Describe each \( G \) equiv class

For a coloring \( c \in C \), define
\[ r = \text{# vertices colored } R \]
\[ w = \text{# vertices colored } W \]
\[ b = \text{# vertices colored } B \]

For \( c \in C \), \( c \) is determined by \( r, w, b \).

obs
\[ r + w + b = 4 \]
\[ r \geq 0, \ w \geq 0, \ b \geq 0 \]

\#sols for \( r, w, b \) is
\[
\binom{4 + 3 - 1}{3 - 1} = \binom{6}{2} = 15
\]

\[ N(G, C) = 15 \]
Solv 2 Use Burnside

<table>
<thead>
<tr>
<th>f ∈ G</th>
<th>C(f) desc</th>
<th></th>
<th>[c(C(f))]</th>
</tr>
</thead>
<tbody>
<tr>
<td>120° cc. rot</td>
<td>b</td>
<td></td>
<td>3 = 9</td>
</tr>
<tr>
<td>120° cc. rot</td>
<td>a, b ∈ {R, w, b}</td>
<td></td>
<td>3² = 9</td>
</tr>
<tr>
<td>180° rot</td>
<td>a, b ∈ {R, w, b}</td>
<td></td>
<td>3² = 9</td>
</tr>
</tbody>
</table>

\[N(\phi, \gamma) = \frac{\sum_{f \in G} |C(f)|}{|G|}\]

= \[\frac{4 \times 3^2 + 4 \times 3^2 + 3 \times 3^2 + 1 \times 3^4}{12}\]

= 15
Ex. For above tetrahedron

Given vertex \( p \)

Given \( p \) dist colors

Let \( C = \text{set of colorings of } X \text{ with these colors} \)

So \( |C| = p^4 \)

Find \( N(G, C) \)

Use Sol 12: By Burnside

\[
N(G, C) = \frac{\sum_{g \in G} |C(g)|}{|G|}
\]

\[
= \frac{4 \times p^2 + 4 \times p^2 + 3 \times p^2 + 1 \times p^4}{12}
\]

\[
= \frac{p^2(p^2 + 11)}{12}
\]

Sol 1 gets complicated
Ex. For the chromatic

Now take \( G = S_4 \) = set of all perms of \( X \)

\( |G| = 4! = 24 \)

Given \( p \geq 1 \), given \( p \) dist colors \( c_1, c_2, \ldots, c_p \)

\( C = \) set of colorings of \( X \) with \( p \) colors

Find \( N(G, C) \)

Sol 1. Desc each equiv class for \( \sim \)

For a coloring \( c \in C \)

For 1st of let

\( n^c_i = \) # vertices colored \( c_i \)

\( C \) is def'd up to \( \sim \) by the sequence

\( n_1, n_2, \ldots, n_p \)

Obs

\( n_1 + n_2 + \ldots + n_p = 4 \)

\( n_i \geq 0 \) \hspace{1cm} 1 \leq i \leq p \)

# such are

\[
\binom{4 + p - 1}{p - 1} = \binom{p + 3}{4}
\]

\( N(G, C) = \binom{p + 3}{4} \)
So | 2
Use Burnside

describe the elements of $G$,

\[ e \Rightarrow f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \]

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \]

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \]

\[ 1 \rightarrow 4 \rightarrow 3 \]

\[ \vartriangle \]

\[ \begin{array}{c|c}
\text{cycle type} & \#	ext{do of } G \\
\hline
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array} & 3^1 = 3 \\
\begin{array}{c}
\Delta \\
\quad \quad \\
\end{array} & 4 \times 2 = 8 \\
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array} & 3 \\
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array} & (\frac{3}{2}) = 6 \\
\begin{array}{c}
\quad \quad \\
\quad \quad \\
\end{array} & 1 \\
\hline
\end{array} \]
| $\rho \in G$ | $C(\rho)$ desc | $|C(\rho)|$ |
|---|---|---|
| ![Square](image) | $a \ a \ \ a \ \ a$ | $\rho$ |
| ![Triangle](image) | $a \ a \ b \ b$ | $\rho^2$ |
| ![Hexagon](image) | $a \ a \ b \ c \ d \ e$ | $\rho^4$ |

$$N(\rho, c) = \frac{\sum_{\rho \in G} |C(\rho)|}{|G|}$$

$$= \frac{6\rho + 8\rho^2 + 3\rho^2 + 6\rho^3 + \rho^4}{24}$$

$$= \frac{6\rho + 11\rho^2 + 6\rho^3 + \rho^4}{24}$$

$$= \frac{(\rho + 3)(\rho + 2)(\rho + 1)\rho}{24}$$

$$= \binom{\rho + 3}{4}$$
Note

\[ \frac{\rho^2 (\rho^2 + 1)}{12} = \binom{\rho + 3}{\omega} \]

for \( \rho = 1, 2, 3 \) but not in general
Given $X$ a nonempty finite set
say $X = \{1, 2, \ldots, n\}$

Given integer $r \ (1 \leq r \leq n)$

Given mutually distinct $a_1, a_2, \ldots, a_r$ in $X$

let

\[
[a_1, a_2, \ldots, a_r]
\]

"cycle notation"

denote the perm of $X$ that sends

$a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \cdots \rightarrow a_r$

and fixes all other elements of $X$

Call the perm a cycle of order $r$

$a$

$r$-cycle
Exercise: $n = 8$

Let $f = [1 \ 3 \ 2 \ 6]$

Write $f$ in 2-line notation.

Solution:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 2 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$$

Note: Each cycle has multiple cycle notations.

For instance:

$$[1 \ 3 \ 2 \ 6] = [3 \ 2 \ 6 \ 1] = [2 \ 6 \ 1 \ 3] = [6 \ 1 \ 3 \ 2]$$

Note: Cycle of order 1 is just the identity:

$$[6] = I$$

In 12i^2n
Given two cycles:

\[ f = [a_1, a_2, \ldots, a_r] \quad g = [b_1, b_2, \ldots, b_s] \]

Call these cycles **disjoint** whenever

\[ a_i \neq b_j \]

In this case, \( f \) and \( g \) commute:

\[ fog = gof \]

pf: obs

- \( f \) fixes each \( b_1, b_2, \ldots, b_s \)
- \( g \) fixes each \( a_1, a_2, \ldots, a_r \)

For \( x \in X \) show

\[ (fog)(x) = (gof)(x) \]

Case \( x \in \{ a_1, a_2, \ldots, a_r \} \)

- say \( x = a_i \)

\[ fog : a_i \rightarrow a_i, \quad \quad g \rightarrow a_i, \quad \quad f \rightarrow a_i \]

\[ gof : a_i \rightarrow a_i, \quad \quad f \rightarrow a_i, \quad \quad g \rightarrow a_i \]
Case $x \in \{ b_1, b_2, \ldots, b_n \}$

Say $x = b_2$

$fog: b_1 \rightarrow b_{2n} \rightarrow b_2 \leftarrow f \rightarrow g$

$gof: b_1 \rightarrow b_2 \rightarrow b_{2n}$

Case $x \notin \{ a_1, a_2 \}$ $x \notin \{ b_1, b_2, \ldots, b_n \}$

$fog: x \rightarrow x \rightarrow x$

$gof: x \rightarrow x \rightarrow x$

Example. Given any perm $f \notin X$

Say $n = 8$

$f = (\begin{array}{ccccccc}
1 & 2 & 4 & 5 & 6 & 7 & 8 \\
4 & 5 & 3 & 6 & 8 & 7 & 1 & 2
\end{array})$

Express $f$ as a product of disjoint cycles $\uparrow$

with respect to composition
\[
f: \quad 1 \rightarrow 4 \rightarrow 6 \rightarrow 7 \quad 2 \rightarrow 5 \rightarrow 8 \quad 3 \rightarrow 3
\]

\[f = \begin{bmatrix} 1 & 4 & 6 & 7 \end{bmatrix} \circ \begin{bmatrix} 2 & 5 & 8 \end{bmatrix} \circ \begin{bmatrix} 3 \end{bmatrix}\]

"cycle factorization of \(f\)"

In general, each permutation of \(X\) is a product of disjoint cycles.

LEM: Each permutation of \(X\) is a product of disjoint cycles.
Ex. Consider regular 5-gon

\[ X = \{1, 2, 3, 4, 5\} \]

Let \( G = \text{group of symmetries} \)

\[ = \text{dihedral group of order 10} \]

\[ G = \{ \rho, \tau \} \]

where

\( \rho \) is clockwise 72° rot.

\( \tau \) is reflection

For each \( f \in G \) find the cycle factorization

\[ \rho \]

\[ \tau \]
<table>
<thead>
<tr>
<th>$f$</th>
<th>cycle factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$[1][2][3][4][5]$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$[12345]$</td>
</tr>
<tr>
<td>$\rho^2$</td>
<td>$[13524]$</td>
</tr>
<tr>
<td>$\rho^3$</td>
<td>$[14253]$</td>
</tr>
<tr>
<td>$\rho^4$</td>
<td>$[15432]$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>![Diagram of $\tau$]</td>
</tr>
<tr>
<td>------</td>
<td>----------------</td>
</tr>
<tr>
<td>$\rho \tau$</td>
<td>![Diagram of $\rho \tau$]</td>
</tr>
<tr>
<td>$\rho^2 \tau$</td>
<td>![Diagram of $\rho^2 \tau$]</td>
</tr>
<tr>
<td>$\rho^3 \tau$</td>
<td>![Diagram of $\rho^3 \tau$]</td>
</tr>
<tr>
<td>$\rho^4 \tau$</td>
<td>![Diagram of $\rho^4 \tau$]</td>
</tr>
</tbody>
</table>
Given \( X = \{1, 2, \ldots, n\} \)

Given \( \text{perm } f \in X \)

Consider cycle factorization of \( f \)

Define \( \#(f) = \text{the number of cycles in this factorization} \)

**Ex:** For above 5-gon

<table>
<thead>
<tr>
<th>( f )</th>
<th>( #(f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>5</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
</tr>
<tr>
<td>( \rho^2 )</td>
<td>1</td>
</tr>
<tr>
<td>( \rho^3 )</td>
<td>1</td>
</tr>
<tr>
<td>( \rho^4 )</td>
<td>1</td>
</tr>
<tr>
<td>( \tau )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho \tau )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho^2 \tau )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho^3 \tau )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho^4 \tau )</td>
<td>3</td>
</tr>
</tbody>
</table>
Given \( X = \{1, 2, \ldots, n\} \)

Given integer \( p \geq 1 \)

Given \( p \) distinct colors \( c_1, c_2, \ldots, c_p \)

Let \( C = \) set of all colorings of \( X \) with these colors

Given a permutation \( f \) of \( X \)

Find \( |C(f)| \) in terms of \( #(f) \)

\( \text{pts in } X \text{ fixed by } f \)

So \( n \times n \)

\( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 & 2 \end{pmatrix} \)

Find cycle factorization of \( f \)

\[ f = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \circ \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix} \circ \begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} \circ \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} \]

\( #(f) = 9 \)
Describe $\hat{C}(f)$

\[
\begin{array}{c}
\begin{array}{c}
 a \\
 a \\
 a \\
\end{array}
\end{array}
\begin{array}{cccc}
 b & b & c & c \\
& & & d
\end{array}
\]

$a, b, c, d \in \{c_1, c_2, \ldots, c_r\}$

\[
\# \text{ choices for } a, b, c, d \text{ is } p^4 = p^{\#(f)}
\]

So

\[
\left| C(f) \right| = p^{\#(f)}
\]

Cor. Let $X, C$ as above

Given $G$ a perm of $X$ then

\[
N(G, C) = \frac{\sum_{\text{fixed } p} p^{\#(f)}}{|G|}
\]
Given \( X = \{1, 2, \ldots, n\} \)

Given a perm \( f \) of \( X \)

We now associate with \( f \) a polynomial in \( n \) variables

\[ z_1, z_2, \ldots, z_n \]

Consider cycle factorization of \( f \)

For \( k \) taken let

\[ e_k = \# \text{ \( k \)-cycles in this factorization} \]

So

\[ \sum_{k=1}^{n} e_k = n \]

Note

\[ \sum_{k=1}^{n} e_k = \# \text{ total cycles in this factorization} = \#(f) \]

Define

\[ mm(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n} \quad \text{"the monomial of \( f \)}" \]

ex

\( n = 8 \)

\[ f = \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 \end{pmatrix} \]

\[ f = [182] \circ [35] \circ [47] \circ [6] \]

\[ e_k \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \]

\[ e_k \quad 1 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \]

\[ mm(f) = z_1^2 z_2 z_3 \]
Def  Given $X = \{1, 2, \ldots, n\}$

Given $G = \text{perm group on } X$

Define a polynomial $P_G$ in $n$ variables

$z_1, z_2, \ldots, z_n$

by

$$P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \text{mult}(f)}{|G|}$$

Thm  Given $X = \{1, 2, \ldots, n\}$

Given $p \geq 1$

Given $q$ distinct colors $c_1, c_2, \ldots, c_p$

let $C = \text{set of all colorings of } X \text{ with these colors}$

let $G = \text{perm group on } X$

Then

$$N(G, C) = P_G(r, r, \ldots, r)$$
Recall

\[ N(c, c) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|} \]

By def

\[ p_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} mm(f)}{|G|} \]

For \( f \in G \) write

\[ mm(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n} \]

setting \( z_i = p^{f(i)} \), then become

\[ p^{e_1} p^{e_2} \cdots p^{e_n} \]

\[ = p^{e_1 + e_2 + \cdots + e_n} \]

\[ = p^{\#(f)} \]

So

\[ p_G(p, p, \ldots, p) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|} \]

\[ = N(c, c) \]

\[ \square \]
Recall

Fix $X = \{1, 2, \ldots, n\}$, $n \geq 1$

$p \geq 1$

Fix colors $c_1, c_2, \ldots, c_p$

$C = \text{set of all colorings of } X \text{ with these colors}$

Fix $G = \text{perm}_p$ on $X$

Then

$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$

Burnside

$= P_G(p, p, \ldots, p)$

where

$P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$

$\text{mon}(f) = \prod_{k=1}^{n} z_{e_1} \cdot e_2 \cdot \cdots \cdot e_n$

$e_k = \# \text{k-cycles in cycle factorization of } f$

Function $P_G$ called cycle index of $G$. 
Given $c \in C$

For $1 \leq i \leq p$ define

$$n_i = \# \text{vertices in } X \text{ colored } c_i \text{ by } c$$

So

$$n_1 + n_2 + \cdots + n_p = n = |X|$$

$$n_i \geq 0 \quad i = 1, 2, \ldots, p$$

Given a set $E \subseteq E_G$ define

$$C_{n_1, n_2, \ldots, n_p} = \text{ set of colorings of } X \text{ that have exactly}$$

$$n_i \text{ vertices colored } c_i \text{ for } i = 1, 2, \ldots, p$$

Note that $C_{n_1, n_2, \ldots, n_p}$ is $G$-invariant.

Problem

Find

$$N(G, C_{n_1, n_2, \ldots, n_p})$$

**
Sol: We give the generating function for $\mathfrak{X}$.

Introduce variables

$$u_1, u_2, \ldots, u_p$$

$$u_i$$ correspond to color $c_i$ for $i \in \mathbb{Z}/p\mathbb{Z}$

Gen function $f$ is

$$\sum_{n_1, n_2, \ldots, n_p} \mathbb{N}(G, c_1, n_1, \ldots, c_p) \quad u_1^{n_1} u_2^{n_2} \ldots u_p^{n_p}$$

where sum is over all color $n_1, n_2, \ldots, n_p \to \ast$
Theorem (Polya Counting Formula)

With above notation

\[ \mathcal{A} = P_G \left( u_1 + u_2 + \ldots + u_r, \; u_1^2 + u_2^2 + \ldots + u_r^2, \; \ldots, \; u_1^n + u_2^n + \ldots + u_r^n \right) \]

where \( P_G \) = cycle index of \( G \)

pf \((h_{1,2})\)

2 colors \( c_1, c_2 \)

Show

\[ \sum_{n_1 \geq 0, \; n_2 \geq 0, \; n_1 + n_2 = n} N \left( G; c_1^{n_1}, c_2^{n_2} \right) u_1^{n_1} u_2^{n_2} = P_G \left( u_1 + u_2, \; u_1^2 + u_2^2, \; \ldots, \; u_1^n + u_2^n \right) \]
Consider

\[ P_G (u_1 + u_2, u_1 + u_2^2, \ldots, u_1 + u_2^n) \]

as a poly in \( u_1, u_2 \).

For \( n \geq 0 \) and \( n \geq 20 \)

\[
\text{coeff of } u_1^n u_2^n
\]

\[
= \frac{1}{n}
\]

Assume \( n_1 + n_2 = n \), else \( \lambda = 0 \) by constr.

Show

\[ \lambda = N \left( G_1, C_{n_1, n_2} \right) \]

By Burnside

\[ N \left( G, C_{n_1, n_2} \right) = \frac{\sum f \mid C_{n_1, n_2} (f)}{|G|} \]

Recall

\[ P_G (z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \text{mm}(f)}{|G|} \]

For \( f \in G \) show

\[ \text{coeff of } \text{mm}(f) \text{ to } \lambda = \left| \frac{C_{n_1, n_2}(f)}{|G|} \right| \]
\[ m \circ f = e_1 e_2 \cdots e_n \]

\[ e_k = \# \text{k-cycles in cycle factorization of } f \]

Replace

\[ e_1 \rightarrow u_1 + u_2 \]
\[ e_2 \rightarrow u_2^2 + u_2^3 \]
\[ \vdots \]
\[ e_n \rightarrow u_n^2 + u_n^n \]

\[ \text{mult}(f) \text{ becomes} \]

\[ (u_1 + u_2)^{e_1} (u_2^2 + u_2^3)^{e_2} \cdots (u_n^2 + u_n^n)^{e_n} \]

In this polynomial show

\[ \text{coef of } u_1^{n_1} u_2^{n_2} = |C_{n_1, n_2}(f)| \]
\[
\left( \sum_{l_1=0}^{e_1} \left( l_1 \right) u_{l_1} e_{l_1} \right) \left( \sum_{l_2=0}^{e_2} \left( l_2 \right) u_{l_2} e_{l_2} \right) \ldots \left( \sum_{l_n=0}^{e_n} \left( l_n \right) u_{l_n} e_{l_n} \right)
\]

In the above polynomial the exph of \( u_{l_1} u_{l_2} \ldots u_{l_n} \) is

\[
\sum_{l_1, l_2, \ldots, l_n \geq 0} \left( \frac{e_1}{l_1} \right) \left( \frac{e_2}{l_2} \right) \ldots \left( \frac{e_n}{l_n} \right)
\]

\( l_1 + 2l_2 + \ldots + nl_n = \pi_1 \)

\[\text{show me equal } |C_{n_1 n_2}(t)| \]
Find $|C_{n_1,n_2}(4)|$

cycle factorization of $f$:

$e_i$ $\{ \ldots \}$ $1$-cycles

$e_2$ $\{ \ldots \}$ $2$-cycles

$e_3$ $\{ \Delta \Delta \Delta \}$ $3$-cycles

$\vdots$

To find $|C_{n_1,n_2}(4)|$ we construct a coloring $c \in C_{n_1,n_2}(f)$ in two steps.
Step I

decide how many

1-cycles get colored \( c_1 \) (\( e_1 \))

2-cycles \( \ldots \) \( c_2 \) (\( e_2 \))

\( \vdots \)

\( n \)-cycles \( \ldots \) \( c_n \) (\( e_n \))

One choice for each odd \( \frac{e_i}{2} \): \( b \)

\( 0 \leq b_1 \leq e_1 \)
\( 0 \leq b_2 \leq e_2 \)
\( \vdots \)
\( 0 \leq b_n \leq e_n \)

\( b_1 + 2b_2 + \ldots + nb_n = N \)

For each choice \( \frac{N}{2} \) we proceed to step II:

Step II

<table>
<thead>
<tr>
<th>steps</th>
<th>to do</th>
<th>#choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>decide which 1-cycles to color ( c_1 )</td>
<td>(( e_1 ))</td>
</tr>
<tr>
<td>2</td>
<td>( \ldots ) 2-cycles ( \ldots )</td>
<td>(( e_2 ))</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( n )</td>
<td>( \ldots ) ( n )-cycles ( \ldots )</td>
<td>(( e_n ))</td>
</tr>
</tbody>
</table>
So

$$| C_{n_1, n_2} (f) | = \sum \left( \frac{e_1}{x_1} \right) \left( \frac{e_2}{x_2} \right) \cdots \left( \frac{e_n}{x_n} \right)$$

$$0 \leq x_1 \leq e_1$$
$$0 \leq x_2 \leq e_2$$
$$\vdots$$
$$0 \leq x_n \leq e_n$$
$$x_1 \cdot x_2 \cdot \ldots \cdot x_n = ne$$

This proves Polya's theorem with $p=2$.\[\square\]
Recall

\[ X = \{1, 2, \ldots, n\} \]

\[ G = \text{permutation group in } X \]

Cycle index of \( G \) is polynomial

\[ P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \text{mm}(f)}{|G|} \]

For \( f \in G \)

\[ \text{mm}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n} \]

\[ e_k = \# k\text{-cycles in cycle factorization of } f \]

Fix \( p \geq 1 \)

distinct colors \( c_1, c_2, \ldots, c_p \)

corresponding variables \( u_1, u_2, \ldots, u_p \)

Polya theorem: Generating function for colorings of \( X \) up to \( G \)-equivalence

\[ = P_G(u_1^0 + u_2^0 + \ldots + u_p^0, u_1^1 + u_2^1 + u_3^1 + \ldots + u_p^1, \ldots, u_1^p + u_2^p + \ldots + u_p^p) \]
This means the following:

For each monomial

$$u_1^{n_1} u_2^{n_2} \ldots u_p^{n_p}$$

in \( \ast \) the following are the same:

(i) the coefficient of \( u_1^{n_1} u_2^{n_2} \ldots u_p^{n_p} \)

(ii) Up to \( G \)-equivalence, the number of colorings of \( X \) with

\[
\begin{align*}
&n_1 \text{ vertices colored } c_1, \\
&n_2 \text{ vertices colored } c_2, \\
&\vdots \\
&n_p \text{ vertices colored } c_p
\end{align*}
\]
Ex. Consider regular 5-gon $X = \{1, 2, 3, 4, 5\}$

$G = \sigma \rho + \text{symmetries}
= \text{dihedral group of order 10}
= \{ \rho^k \}_{i=0}^9 \cup \{ \rho^i \sigma \}_{i=0}^4$

$\rho = \text{clockwise 72° rotation}$
$\sigma = \text{reflection}$

Find $\rho G$
<table>
<thead>
<tr>
<th>$f$</th>
<th>cycle factorization of $f$</th>
<th>$\text{mm}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{}</td>
<td>$\frac{5}{z_1}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>[\text{pentagon}]</td>
<td>$z_5$</td>
</tr>
<tr>
<td>$\rho^2$</td>
<td>[\text{pentagon}]</td>
<td>$z_5$</td>
</tr>
<tr>
<td>$\rho^3$</td>
<td>[\text{pentagon}]</td>
<td>$z_5$</td>
</tr>
<tr>
<td>$\rho^4$</td>
<td>[\text{pentagon}]</td>
<td>$z_5$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>[\text{pentagon}]</td>
<td>$z_1^2 z_2$</td>
</tr>
<tr>
<td>$\rho \tau$</td>
<td>[\text{pentagon}]</td>
<td>$z_1^2 z_2$</td>
</tr>
<tr>
<td>$\rho^2 \tau$</td>
<td>[\text{pentagon}]</td>
<td>$z_1^2 z_2$</td>
</tr>
<tr>
<td>$\rho^3 \tau$</td>
<td>[\text{pentagon}]</td>
<td>$z_1^2 z_2$</td>
</tr>
<tr>
<td>$\rho^4 \tau$</td>
<td>[\text{pentagon}]</td>
<td>$z_1^2 z_2$</td>
</tr>
</tbody>
</table>

$$p_6 \left( \frac{z_1}{z_1} + \frac{z_2}{z_2} + \frac{z_3}{z_3} + \frac{z_4}{z_4} + \frac{z_5}{z_5} \right) = \frac{5}{z_1 + 9 z_2^2 + 4 z_5}$$
Take \( p = 3 \) colors

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>Red</th>
<th>White</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_i )</td>
<td>( r )</td>
<td>( w )</td>
<td>( b )</td>
</tr>
</tbody>
</table>

Find some Polya generating function

\[
P_6 (r+w+b, r^2 w + b^2, r^3 w^2 + b^3, r^4 w^3 + b^4, r^5 w^5 + b^5)
\]

\[
= \frac{(r+w+b)^5 + 5(r+w+b)(r^2 w + b^2)^2 + 4(r^5 w^5 + b^5)}{10}
\]

Up to \( G \)-equivalence, find the number of ways to color the vertices in \( X \) such that

- 2 vertices colored Red,
- 2 vertices colored White,
- 1 vertex colored Blue.

\textbf{Sols.} In Polyga generating function

\[
\text{cof of } r^2 w^2 b = \frac{\left( \frac{5}{2 \pi i} \right) + 5 \times 2}{10}
\]

\[
= 4
\]
The colorings are:

1. B
   - R
   - W
   - W

2. B
   - W
   - R

3. B
   - W
   - R
   - W

4. B
   - W
   - R
   - R
Consider regular 8-gon

\[ X = \{1, 2, \ldots, 8\} \]

\[ G = \text{group of symmetries} \]
\[ = \text{Dihedral group order 16} \]
\[ = \mathbb{Z}_8 \times \mathbb{Z}_2 \cup \mathbb{Z}_4 \times \mathbb{Z}_2 \]

\[ \rho = 90^\circ \text{ clockwise rot} \]

\[ \tau = \text{reflection} \]

Find \( P_6 \)
<table>
<thead>
<tr>
<th>( f )</th>
<th>cycle factorization of ( f )</th>
<th>( \text{mm}(f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_1^e )</td>
</tr>
<tr>
<td>( p )</td>
<td>[ \bigcirc ]</td>
<td>( \mathbb{Z}_2^g )</td>
</tr>
<tr>
<td>( p^2 )</td>
<td>[ \square ]</td>
<td>( \mathbb{Z}_2^2 )</td>
</tr>
<tr>
<td>( p^3 )</td>
<td>[ \bigcirc ]</td>
<td>( \mathbb{Z}_2^g )</td>
</tr>
<tr>
<td>( p^4 )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_2^4 )</td>
</tr>
<tr>
<td>( p^5 )</td>
<td>[ \bigcirc ]</td>
<td>( \mathbb{Z}_2^g )</td>
</tr>
<tr>
<td>( p^6 )</td>
<td>[ \square ]</td>
<td>( \mathbb{Z}_2^2 )</td>
</tr>
<tr>
<td>( p^7 )</td>
<td>[ \bigcirc ]</td>
<td>( \mathbb{Z}_2^g )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_2^4 )</td>
</tr>
<tr>
<td>( p \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_1^4 )</td>
</tr>
<tr>
<td>( p^2 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_1^2 \mathbb{Z}_2^3 )</td>
</tr>
<tr>
<td>( p^3 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_2^4 )</td>
</tr>
<tr>
<td>( p^4 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_1^2 \mathbb{Z}_2^3 )</td>
</tr>
<tr>
<td>( p^5 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_2^4 )</td>
</tr>
<tr>
<td>( p^6 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_1^2 \mathbb{Z}_2^3 )</td>
</tr>
<tr>
<td>( p^7 \tau )</td>
<td>[ \cdots ]</td>
<td>( \mathbb{Z}_2^4 )</td>
</tr>
</tbody>
</table>

\[
\rho_G(z_1, \ldots, z_8) = \frac{z_1^8 + 4z_1^2 z_2^3 + 5z_2^4 + 2z_4^2 + 4z_8}{16}
\]
Up to $G$-equiv, how many ways to color the vertices in $X$ with colors Red, white, Blue?

Sol

Set $Z_i = 3$ for $1 \leq i \leq 8$

$$p_G(3, 3, \ldots, 3) = \frac{3^8 + 4 \cdot 3^5 + 5 \cdot 3^4 + 2 \cdot 3^2 + 4 \cdot 3}{16}$$

$$= 166 \times 3$$

$$= 498$$
Ex. Let $n$ be a prime number.

Fix $p \geq 1$, Dist. Colors $c_1, \ldots, c_p$.

How many different necklaces can be made using $n$ beads with alone colors?

Let $X$ be set of bead locations $= \{1, 2, \ldots, n\}$.

Here symmetry $\pi$ is

$G = \text{dihedral}_n \times \text{order } \mathbb{Z}_n$

$= \{ \rho^i \}_{i=0}^{n-1} \cup \{ \rho^i \circ \tau \}_{i=0}^{n-1}$

$\rho = \text{clockwise rotation} \text{ thru } \frac{360}{n} \text{ degrees}$

$\tau = \text{reflection}$

Find $P_6$

<table>
<thead>
<tr>
<th>$f$</th>
<th>Cycle factorization</th>
<th>$mm(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$\cdots$</td>
<td>$\mathbb{Z}_n$</td>
</tr>
<tr>
<td>$\rho^i$</td>
<td>$\bigcirc$</td>
<td>$\mathbb{Z}_n$</td>
</tr>
<tr>
<td>$\rho^i \circ \tau$</td>
<td>$\xrightarrow{i}$</td>
<td>$\mathbb{Z}_1 \times \mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

$P_6(z_1, \ldots, z_n) = \frac{z_1^n + (n-1) \mathbb{Z}_n + n \mathbb{Z}_1 \mathbb{Z}_2^{\frac{2}{n}}}{\mathbb{Z}_n}$
# necklaces

= \# \text{ways to color } X \text{ with colors } c_1, c_2, \ldots, c_p \\
\text{(up to } G\text{-equivalence)}

= P_G(p, p, \ldots, p)

= \frac{p^n + (n-1)p + np^{\frac{\Delta H}{2}}}{2n}

Now for \( p = 2 \) colors Red, Blue

Find the Polya gen function

<table>
<thead>
<tr>
<th>\xi_i</th>
<th>Red</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>\omega</td>
<td>r</td>
<td>b</td>
</tr>
</tbody>
</table>

Polya Gen function

\[ P_G(r + b, r^2 + b^2, \ldots, r^n + b^n) \]

= \frac{(r+b)^n + (n-1)(r^n+b^n) + n(r+b)(r^2+b^2)^{\frac{\Delta H}{2}}}{2n}
Ex. Find the cycle index for the dihedral group $D_{2p}$ if $p$ is prime.

$(9p$ has order $4p)$

Sol. View $G$ as corner-symmetry group of regular $2p$-gon.

$G = \left\{ p^i \zeta_3^j \mid i = 0, \ldots, n \right\} \cup \left\{ p^i \rho \zeta_3^j \mid i = 0, \ldots, n \right\}$

$\rho$: clockwise rot thru $\frac{360}{n}$

$\tau$: reflection
<table>
<thead>
<tr>
<th>$f$</th>
<th>cycle factorization of $f$</th>
<th>$\text{mm}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$\overbrace{\cdots \cdot \cdot}^{n}$</td>
<td>$\mathbb{Z}_1$</td>
</tr>
<tr>
<td>$p^i$</td>
<td>$n$-cycle</td>
<td>$\mathbb{Z}_n$</td>
</tr>
<tr>
<td>$i \text{ odd}$</td>
<td>$\leq i &lt; n$</td>
<td>$\mathbb{Z}_i$</td>
</tr>
<tr>
<td>$p^i$</td>
<td>$\circ \circ$</td>
<td>$\mathbb{Z}_1 \times \mathbb{Z}_2^{r^r}$</td>
</tr>
<tr>
<td>$i \text{ even}$</td>
<td>$\leq i &lt; n$</td>
<td>$\mathbb{Z}_2^r$</td>
</tr>
<tr>
<td>$p^i \cdot r$</td>
<td>$\leq i \leq n-1$</td>
<td>$\mathbb{Z}_1 \cdot \mathbb{Z}_2^r$</td>
</tr>
<tr>
<td>$p^i \cdot r$</td>
<td>$i \text{ odd}$</td>
<td>$\mathbb{Z}_2^r$</td>
</tr>
<tr>
<td>$0 \leq i &lt; n$</td>
<td>$\mathbb{Z}_1 \cdot \mathbb{Z}_2^r$</td>
<td></td>
</tr>
</tbody>
</table>

\[ p_6(z_1, \ldots, z_n) = \frac{z_1^{2r} + p \cdot z_{2r} + (p - 1) \cdot z_r^{2r} + p \cdot z_1^{2} \cdot z_2^{p^r} + p \cdot z_2^p}{4p} \]
Ex. Find the Polya gen function for the number of\ndifferent necklaces that can be
made with $2p$ beads and two colors Red, Blue
$p = \text{prime}$

Sol. In prev problem replace

$$Z_i = r^i + b^i \quad 1 \leq i \leq 2p$$

Polya Gen Function is

$$\frac{(r+b)^{2p} + p(r^{2p} + b^{2p}) + (p)(r^p+b^p)^2 + p(r+b)(r^2+b^2)^p + p(r^2b^2)^p}{4p}$$

The End ♡