Ch 4 Generating Permutations and Combinations

4.1 Generating permutations

Probs: List all the permutations of \( \{1, 2, \ldots, n\} \)

ex: \( n = 3 \)

\[
\begin{align*}
123 \\
132 \\
213 \\
231 \\
312 \\
321
\end{align*}
\]

For large \( n \), hard to keep track!

We now present a listing method with nice properties

\( n = 2 \)

\[
\begin{align*}
12 \\
21
\end{align*}
\]

\( n = 3 \) In the \( n = 2 \) list, insert 3 in all possible ways as follows:

\[
\begin{align*}
123 \\
132 \\
312 \\
321 \\
231 \\
213
\end{align*}
\]
$n=4$

\[
\begin{align*}
1 & 2 & 3 & 4 & \rightarrow & 1243 & \rightarrow & 1234 & \rightarrow & 4123 \\
3 & 1 & 2 & 4 & \leftarrow & 3124 & \leftarrow & 1324 & \leftarrow & 4132 \\
3 & 2 & 1 & 4 & \downarrow & 3214 & \downarrow & 3241 & \downarrow & 4321 \\
2 & 3 & 1 & 4 & \rightarrow & 2314 & \rightarrow & 2341 & \rightarrow & 4231 \\
2 & 1 & 3 & 4 & \leftarrow & 2134 & \leftarrow & 2143 & \leftarrow & 4213 \\
\end{align*}
\]

$n=5$ similar

For $n=2$, the list of permutations of $\{1,2,\ldots,n\}$ is obtained from the list of permutations of $\{1,2,\ldots,n-1\}$ by inserting "n" in all possible ways, as shown above.
Nice properties

Given a permutation \( a_1 a_2 \ldots a_n \) of \( \{1, 2, \ldots, n\} \)

A transposition of \( a_1 a_2 \ldots a_n \) switches adjacent terms

\[
\begin{array}{c}
\downarrow & \downarrow & \downarrow & \downarrow \\
 a_1 a_2 & a_1 a_n & \ldots & a_n \\
\end{array}
\]

"the transposition"

\[
\begin{array}{c}
\downarrow & \downarrow & \downarrow & \downarrow \\
 a_1 a_2 & a_3 a_4 & \ldots & a_n \\
\end{array}
\]

\( a_1 a_2 \ldots a_n \) has \( n-1 \) transpositions

\( \text{ex} \ n=4 \)

\[
\begin{array}{c}
1423 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1243 & 1423 & 1432 \\
\end{array}
\]

1st trans
2nd trans
3rd trans

* In our list of perms of \( \{1, 2, \ldots, n\} \)
each perm in list is a transposition of
the preceding perm. Also, the first
perm is a transposition of last perm.
ex n = 3

\[
\begin{array}{c}
\text{start} \rightarrow \quad 1 \quad 2 \quad 3 \\
\quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
2 \quad 1 \quad 3 \\
\quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
3 \quad 2 \quad 1 \\
\end{array}
\]

\[a = 1\text{st transposition}\]
\[b = 2\text{nd transposition}\]
Listing order follows path

\[ a = \text{kit transposition} \]
\[ b = 2\text{nd} \]
\[ c = 3\text{d} \]

View as polytope in 3 dimensions
Problem: Given perm \( a_1 a_2 \ldots a_n \) of \{1, 2, \ldots, n\}, what is the next perm in our list?

- Appears difficult to answer without reproducing the entire list up to \( a_1 a_2 \ldots a_n \).

- To remove this difficulty, we add information as follows:

- For \( i \) \in \{1, 2, \ldots, n\} assign a direction to \( a_i \):
  \[
  a_k \rightarrow a_i \leftarrow a_k
  \]

- For \( i \) \in \{1, 2, \ldots, n\} called mobile if its arrow points to an adjacent smaller member.

\( n=5 \)

\[
\begin{array}{cccc}
\rightarrow & \leftarrow & \leftarrow & \leftarrow \\
4 & 5 & 1 & 3 & 2 \\
\uparrow & \nearrow & & & \\
& mobile & & & \\
\end{array}
\]
The following algorithm generates all the permutations of \( \{1, 2, \ldots, n\} \) in the order that we discussed earlier:

1. Start with \( 1 \ 2 \ \ldots \ n \)

While there exists a mobile integer do:

1. Find the largest mobile integer \( m \)
2. Switch \( m \) with its adjacent integer it points to
3. Change the direction of each integer greater than \( m \)
4.2 Inversions of Permutations

For an integer $n \geq 1$

there are $n!$ perms of $\{1,2,\ldots,n\}$

Consider Cartesian product

$$\{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-2\} \times \cdots \times \{0,1,2\} \times \{0,1\} \times \{0\}$$

(*)

The set (*) has $n!$ elements.

Next goal: display a bijection between (*) and the set of all perms of $\{1,2,\ldots,n\}$

Given a perm $a_1, a_2, \ldots, a_n$ of $\{1,2,\ldots,n\}$

an inversion of this perm is an ordered pair $(a_k, a_l)$

such that

$k < l$ and $a_k > a_l$

"$a_k$ and $a_l$ are out of order"
Given a permutation \( a_1a_2\ldots a_n \) of \( 1, 2, \ldots, n \)

For \( 1 \leq i \leq n \) let

\[
b_i = \# \text{ of inversions that have } i \text{ as } i^{th} \text{ term}
\]

\[
= \# \text{ of elements among } a_1, a_2, \ldots, a_n \text{ that are larger than } i \text{ and appear to left of } i.
\]

The sequence \( (b_1, b_2, \ldots, b_n) \) is called the inversion sequence.

For \( a_1a_2\ldots a_n \).

Note \( b_1 + b_2 + \ldots + b_n = \) total number of inversions for \( a_1a_2\ldots a_n \).

\[\text{ex} \quad n=4 \quad \text{perm} \ 4132 \]

\[
\begin{array}{c|cccc}
  i & 1 & 2 & 3 & 4 \\
  \hline
  b_i & 1 & 2 & 1 & 0
\end{array}
\]

\[1 + 2 + 1 + 0 = 4 = \# \text{ inversions for } 4132\]
LEM Given a perm \((a_1, a_2, \ldots, a_n)\) with inversion sequence \((b_1, b_2, \ldots, b_n)\)

then

\[
\begin{align*}
0 &\leq b_1 \leq n-1 \\
0 &\leq b_2 \leq n-2 \\
0 &\leq b_3 \leq n-3 \\
& \vdots \\
0 &\leq b_{n-2} \leq 1 \\
0 &\leq b_{n-1} \leq 1 \\
o &\geq b_n
\end{align*}
\]

pf For \(1 \leq i \leq n\) show

\[0 \leq b_i \leq n-i\]

Call the perm \((a_1, a_2, \ldots, a_n)\)

\[b_i = \# \text{ elements among } a_1, a_2, \ldots, a_n \text{ that are larger than } i
\text{ and to left of } i
\]

\[\leq \# \text{ elements among } a_1, a_2, \ldots, a_n \text{ that are larger than } i
\text{ \underline{choices} } a_1, a_2, \ldots, a_n, n
\]

\[\leq n-i
\]
We can recover a perm of \( 1, 2, \ldots, n \) from its inversion sequence as follows.

Example: \( n = 6 \)

<table>
<thead>
<tr>
<th>1 2 3 4 5 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 0 2 1 1 0</td>
</tr>
</tbody>
</table>

Find any perm and answer \#6

**Method 1**

6
6 5
6 4 5
6 4 3 5
2 6 4 3 5
2 6 4 3 4 5

**Method 2**

---

1
2 1
2 3 1
2 4 3 1
2 4 3 1 5
2 6 4 3 1 5
In summary we have

**Theorem**: Given an integer \( n \geq 2 \),

the function which sends a permutation of \( \{1,2,...,n\} \) to its

inversion sequence is a bijection from the set

of all permutations of \( \{1,2,...,n\} \) to the set

\[
\{0,1,...,n-2\} \times \{0,1,...,n-2\} \times \cdots \times \{0,1,2\} \times \{0\} \times \{0\}
\]

\[(\ast)\]

**Proof**: Cardinality of set \((\ast)\) is \( n! \),

\[
= \text{\# perms of } \{1,2,...,n\}
\]

So suffices to show function is 1-1.

The function is 1-1 since each perm of \( \{1,2,...,n\} \) is determined

by its inversion sequence.

\(\square\)
Given a perm \( a_1 a_2 \ldots a_n \) of \( \{1, 2, \ldots, n\} \)

with inversion sequence \( (b_1, b_2, \ldots, b_n) \).

Recall

\[ b_1 + b_2 + \ldots + b_n = \text{total number of inversions} \]

"inversion number" = "length"

The inversion number is equal to the minimum number of transpositions required to bring \( a_1 a_2 \ldots a_n \) to \( 1 2 3 \ldots n \).

Reason:

Let \( l = \text{inversion number of } a_1 a_2 \ldots a_n \)

At most \( l \) transpositions required:

- Use \( b_1 \) transpositions to move "1" to position 1.
  
  \[ 1 \text{xx} \ldots \text{xx} \]

- Use \( b_2 \) transpositions to move "2" to position 2.
  
  \[ \text{1xx} \ldots \text{xx} 2 \]

- Use \( b_n \) transpositions to move "n" to position n.
  
  \[ \text{1xx} \ldots \text{xx} n \]

At least \( l \) transpositions required: each transposition changes the inversion number by \( \pm 1 \).
ex  \( n=6 \)  \( \text{perm} \) 624135

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_i )</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Inversion sequence**

**Inversion number** = \( 3 + 1 + 2 + 1 + 1 + 0 \)

= 8

Bring 624135 to 123456 using 8 transpositions:

<table>
<thead>
<tr>
<th>( \text{perm} )</th>
<th>( \text{to do} )</th>
<th>( # \text{transpositions used} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>624135</td>
<td>move &quot;1&quot;</td>
<td>3</td>
</tr>
<tr>
<td>( \times ) 621435</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 612435</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 162435</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 126435</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 126345</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 123645</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 123465</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times ) 123456</td>
<td></td>
<td></td>
</tr>
<tr>
<td>123456</td>
<td>move &quot;6&quot;</td>
<td>0</td>
</tr>
</tbody>
</table>
4.3 Generating Combinations

Problem: Given a set $S$ with $|S| = n$

List all the subsets of $S$

Ex $n=3$ $S = \{1, 2, 3\}$

One list is

$\emptyset \, \{1\} \, \{2\} \, \{3\} \, \{1, 2\} \, \{1, 3\} \, \{2, 3\} \, \{1, 2, 3\}$

For large $n$ gets complicated

We now present a listing method with nice properties

From now on take

$S = \{x_{n-1}, x_{n-2}, \ldots, x_2, x_1, x_0\}$

Step 1

We identify each subset $\Omega$ of $S$ with a sequence

$a_n a_{n-1} \ldots a_1 a_0$ of zeros and ones.

For $0 \leq i \leq n-1$

$$a_i = \begin{cases} 1 & \text{if } x_i \in \Omega \\ 0 & \text{if } x_i \notin \Omega \end{cases}$$
Exercise \( n = 6 \)

\[ S = \{ x_5, x_4, x_3, x_2, x_1, k_0 \} \]

Take

\[ \Omega = \{ x_0, x_1, x_2 \} \]

<table>
<thead>
<tr>
<th>( x_5 )</th>
<th>( x_4 )</th>
<th>( x_3 )</th>
<th>( x_2 )</th>
<th>( x_1 )</th>
<th>( x_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 2**

View each sequence of zeros and ones as the "base 2" representation of a non-negative integer.

Example: The sequence 0 1 0 0 1 1 0 corresponds to the integer

\[
0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^0 \\
= 16 + 4 + 2 \\
= 22
\]
ex. Find the base 2 representation of 57

Sol.

Recall

<table>
<thead>
<tr>
<th>2^0</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

take out 32

\[
\begin{array}{c}
57 \\
-32 \\
-32 \\
25 \\
-16 \\
-16 \\
9 \\
-9 \\
-9 \\
8
\end{array}
\]

answer = 111001
Step 3

the list

\[ 0, 1, 2, 3, \ldots, 2^n - 1 \]

de of integers, each expressed in base 2,
effectively gives all 2^n subsets
of an n-element set

"squashed order"

\[ \text{ex } n=3 \]

<table>
<thead>
<tr>
<th>m</th>
<th>m in base 2</th>
<th>correspond subset of {x_2, x_1, x_0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
<td>{x_0}</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
<td>{x_1}</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
<td>{x_1, x_0}</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>{x_2}</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>{x_2, x_0}</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>{x_2, x_1}</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>{x_2, x_1, x_0}</td>
</tr>
</tbody>
</table>
ex \ F_n \ n=7

Consider subset

\{ x_5, x_4, x_2, x_1, x_0 \}

f

S = \{ x_5, x_4, x_3, x_2, x_1, x_0 \}

Find the next subset in the squashed order.

Sol. Convert to base 2:

\[
\begin{array}{cccccccc}
  x_6 & x_5 & x_4 & x_3 & x_2 & x_1 & x_0 \\
  0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

add 1:

\[
\begin{array}{cccccccc}
  0 & 1 & 1 & 0 & 1 & 1 & 1 \\
+ & 0 & 0 \\
\hline
  0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

Convert back:

\[
\{ x_5, x_4, x_3 \}
\]
We just showed how to list the subsets of an n-element set.

We now give an alternative approach using Gray codes.

**Problem** For n = 1, list all the n-tuples of zeros and ones such that each n-tuple differs from previous one in exactly one coordinate.

Such a list is called a Gray code of order n.

**Example** n = 3

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Definition** A Gray code is cyclic whenever 1st term and last term differ in exactly one coordinate.
Geometric interpretation of Gray codes

For an integer \( n \geq 1 \) we define a graph called the \( n \)-cube.

- The vertex set \( X \) consists of all \( n \)-tuples of 0's and 1's. So \( |X| = 2^n \).
- Two vertices are declared adjacent whenever they differ in exactly one coordinate.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n )-cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 — 1</td>
</tr>
<tr>
<td>2</td>
<td>( \begin{array}{c} 10 — 11 \ 00 — 01 \end{array} )</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{array}{c} 110 — 111 \ 010 — 011 \ 000 — 001 \ 100 — 101 \end{array} )</td>
</tr>
</tbody>
</table>
Observe

1. A Gray Code of order $n$ is a path through the $n$-cube that visits each vertex exactly once.

2. The Gray Code is cyclic, where the last vertex is adjacent to the first vertex.

Example: $n = 3$

```
110 → 111 → 101 → 001 → 011 → 010 → 110
```

0 = start

cyclic

Gray Code
We now describe a special cyclic Gray code called the Reflected Gray Code.

To display the reflected Gray code of order 4,

<table>
<thead>
<tr>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Algorithm to obtain the reflected Gray code of order $n$:

- Reflected Gray code of order $n=1$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- For $n \geq 2$, the reflected Gray code of order $n$ is obtained from the reflected Gray code of order $n-1$ by creating 2 copies, with the 2nd copy in inverted order.
  - In each term in Copy 1, add a leading 0.
  - In each term in Copy 2, add a leading 1.
Lecture 11  Friday  Sept 28

We continue to discuss the reflected Gray codes.

Problem: Given a term in the reflected Gray code of order $n$:

$g_0 \ldots g_{2^n} g_0$

(i) What is next term?
(ii) What is preceding term?

If we list the codewords for small values of $n$,

the following pattern emerges.

For (i), (ii) we just need to specify which codeword to change.

(i) $F_n \sum_{i=0}^{n-1} g_i$ even change $g_0$

(ii) $F_n \sum_{i=0}^{n-1} g_i$ odd change $g_n$ for the unique integer $2 \leq 1$ such that $g_{n-1} = 1$ and each of $g_0, g_1, \ldots, g_{2^n}$ is a 0.

\[ \ldots \ast \ast \square 100 \ldots 0 \]

-> change
(i) $F_n \sum_{i=0}^{n-1} g_i$ odd, change $g_0$

For $\sum_{i=0}^{2n-1} g_i$ even, change $g_0$ for some unique integer $i \geq 1$ such that $g_{i-1} = 1$ and each of $g_0, g_1, \ldots, g_{i-2} = 0$

Ex: Consider reflected Gray code of order 8

For the term

<table>
<thead>
<tr>
<th></th>
<th>7 6 5 4 3 2 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_i$</td>
<td>1 0 1 0 0 1 1 0</td>
</tr>
</tbody>
</table>

(i) Find next term

(ii) Find preceding term

Note $\sum g_i = 4$ is even

(iii) Change $g_0$:

ans = 1 0 1 0 0 1 1 1

(iii) Find $i$ such that $g_{i-1} = 0$ and $g_i = 1$

\[ g_0 g_1 \ldots g_{i-1} = \square 10 \ldots 00 \]

$g_{i-2} = 1$

\[ \uparrow \text{change} \]

ans = 1 0 1 0 0 0 1 0
4.4 Generating r-subsets

Problem
Given a finite set \( S \).
Given integer \( r \) s.t. \( 1 \leq r \leq |S| \).
List all the \( r \)-subsets of \( S \).

Solution 1:
List all the \( r \)-subsets of \( S \) using the method of Section 4.2.
Discard each term that does not have cardinality \( r \).

Solution 2:
Use lexicographical order as described below.

Example
\( |S| = 26 \), \( r = 3 \)

View \( S \) as the letters of the alphabet:
\( S = \{a, b, c, \ldots \} \).

View each \( 3 \)-subset of \( S \) as a word of length 3 and list them in alphabetical order:

\[
\begin{align*}
XYZ & \quad \text{ Lex order} \\
abc & \quad \text{ Lex order} \\
abd & \quad \text{ Lex order} \\
abc & \quad \text{ Lex order} \\
abz & \quad \text{ Lex order} \\
acd & \quad \text{ Lex order} \\
ace & \quad \text{ Lex order} \\
acz & \quad \text{ Lex order} \\
ace & \quad \text{ Lex order}
\end{align*}
\]
Ex. Index order, list all the 3-sets of \{1, 2, 3, 4, 5\}.

Sol. Consider alphabet with "letters" 1 < 2 < 3 < 4 < 5.

In alphabetical order, list all words of form \(xyz\) such that 1 < 4 < z < 5.

123
124
125
134
135
145
234
235
245
345
Ex. Consider the 5-subset of 
\{1, 2, 3, 4, 5, 6, 7, 8, 9\}

in Lex order.

What is position \(1\) \(3\) \(4\) \(7\) \(8\) \(\text{ (X)}\)

If \((\text{X})\) is the \(m\)th term, what is \(m\)?

[start counting with \(m = 1\)]

Sol. Strategy: count the terms that come after \((\text{X})\).

<table>
<thead>
<tr>
<th>types of 5-subsets</th>
<th># choices</th>
</tr>
</thead>
<tbody>
<tr>
<td>No rest. abcde (1\leq a &lt; b &lt; c &lt; d &lt; e \leq 9)</td>
<td>(\binom{9}{5})</td>
</tr>
<tr>
<td>abcde (2 \leq a &lt; b &lt; c &lt; d &lt; e \leq 9)</td>
<td>(\binom{8}{5})</td>
</tr>
<tr>
<td>(1\text{bcd}e) (4 \leq b &lt; c &lt; d &lt; e \leq 9)</td>
<td>(\binom{6}{4})</td>
</tr>
<tr>
<td>(1\text{cde}) (5 \leq c &lt; d &lt; e \leq 9)</td>
<td>(\binom{5}{3})</td>
</tr>
<tr>
<td>({1, 3, 4} \text{ac}) (8 \leq a &lt; c \leq 9)</td>
<td>(\binom{2}{2}) = 1</td>
</tr>
<tr>
<td>({1, 3, 4} \text{ec}) (9 \leq e \leq 9)</td>
<td>(\binom{1}{1}) = 1</td>
</tr>
</tbody>
</table>

\(m = \binom{9}{5} - \binom{8}{5} - \binom{6}{4} - \binom{5}{3} - \binom{2}{2} - \binom{1}{1}\)
Then for $1 \leq r \leq n$ consider $r$-subsets of $\{1, 2, \ldots, n\}$ in lexicographic order.

The pos of the $r$-subset $\{a_1, a_2, \ldots, a_r\}$ where $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ is

$$
\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r} - \cdots - \binom{n-a_{r-1}}{r} - \binom{n-a_r}{1}
$$

pf. Same idea as in prev example. \qed
Problem

Given a finite set $S$

Given integer $r$ with $0 \leq r \leq |S|$

List all the $r$-permutations of $S$

Sol:

First list the $r$-subsets of $S$ in lexicographic order.

For each subset, list all the permutations of its elements using the method of Section 7.1.

Ex

List all the 3-perms of $\{1, 2, 3, 4, 5\}$.

Sol:

First list 3-subsets:

- $\{1, 2, 3\}$
- $\{1, 2, 4\}$
- $\{1, 2, 5\}$
- $\{1, 3, 4\}$
- $\{1, 3, 5\}$
- $\{1, 4, 5\}$
- $\{2, 3, 4\}$
- $\{2, 3, 5\}$
- $\{2, 4, 5\}$
- $\{3, 4, 5\}$

Expand:

- $123$ $132$ $213$ $231$ $312$ $321$
- $124$ $142$ $214$ $241$ $412$ $421$
- $134$ $143$ $314$ $341$ $431$ $432$
- $234$ $243$ $324$ $342$ $423$ $432$
4.5 Partial Orders and Equivalence Relations

Consider a set \( X \)

Consider the Cartesian product

\[ X \times X = \{ (a,b) \mid a, b \in X \} \]

A subset

\[ R \subseteq X \times X \]

is called a relation on \( X \)

For \( a, b \in X \) write

\[ a \mathbin{R} b \quad \text{whereas} \quad (a, b) \in R \quad \text{"}a\text{ is related to } b\text{"

\[ a \not\mathbin{R} b \quad \ldots \quad (a, b) \notin R \quad \text{"}a\text{ is not related to } b\text{"

Ex 1 \( X = \) set of integers

\[ \forall a, b \in X \]

\[ a \mathbin{R} b \quad \text{whereas} \quad a - b \text{ is even} \]

Ex 2 \( X = \) set of all subsets of \( \{ 1, 2, \ldots, n \} \)

\[ \forall a, b \in X \]

\[ a \mathbin{R} b \quad \text{whereas} \quad a \subseteq b \]
Ex 3 \( X = \{1, 2, \ldots, 100\} \)

\[ \forall a, b \in X \]

\[ a R b \quad \text{ whenever } a < b \]

We now list some important conditions on relations.

Given a relation \( R \) on a set \( X \),

<table>
<thead>
<tr>
<th>Condition</th>
<th>Meaning</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflexive</td>
<td>( x R x ) (for all ( x \in X ))</td>
<td>1, 2</td>
</tr>
<tr>
<td>Irreflexive</td>
<td>( x \not R x ) (for all ( x \in X ))</td>
<td>3</td>
</tr>
<tr>
<td>Symmetric</td>
<td>( x R y ) implies ( y R x ) (for all ( x, y \in X ))</td>
<td>1</td>
</tr>
<tr>
<td>Antisymmetric</td>
<td>For distinct ( x, y \in X ), [ x R y ] implies ( y \not R x )</td>
<td>2, 3</td>
</tr>
<tr>
<td>Transitive</td>
<td>( x R y ) and ( y R z ) implies ( x R z ) (for all ( x, y, z \in X ))</td>
<td>1, 2, 3</td>
</tr>
</tbody>
</table>
Def Given a relation R on a set X,

- Call R a **partial order** whenever R is reflexive, antisymmetric, transitive.

- Call R a **strict partial order** whenever R is irreflexive, antisymmetric, transitive.

- The set X together with a partial order on X is called a **partially ordered set** (poset).

- For \( x, y \in X \), call \( x \sim y \) **comparable** whenever \( xRy \) or \( yRx \).

- Call \( x \not\sim y \) **incomparable** whenever \( x \not\sim y \) and \( y \not\sim x \).

- Call R a **total order** whenever R is a partial order and \( x \sim y \) is comparable for all \( x, y \in X \).
Ex. Given integer $n \geq 1$

$X = \{1, 2, \ldots, n\}$

Given permutation $a_1, a_2, \ldots, a_n$ of $X$

Define a relation $R$ on $X$ by:

$\forall \ x, y \in X$

$xRy \iff \ x = y$ or $x$ comes before $y$ among $a_1, a_2, \ldots, a_n$

Then $R$ is a total order (check)

Ex. Given integer $n \geq 1$

$X = \{1, 2, \ldots, n\}$

Given total order $R$ on $X$

Call an element $x \in X$ **minimal** whenever there does not exist $y \in X$ s.t. $y \neq x$ and $yRx$

Observe

$X$ has **unique** minimal element $a_1$

$X \setminus \{a_1\} \ni a_2$

$X \setminus \{a_1, a_2\} \ni a_3$

$\vdots$

Then $a_1, a_2, \ldots, a_n$ is permutation of $X$. 
The previous 2 examples show:

Then, given integer \( n \geq 1 \)

\[ X = \{ x_1, \ldots, x_n \} \]

There is a 1-1 correspondence between the total orders on \( X \)
and the permutations of \( X \).

---

A generic partial order on a set \( X \) is usually denoted \( \leq \).

For \( x, y \in X \) write \( x \leq y \) whenever \( x + y \) and \( x \neq y \).

Def: Given a set \( X \) with partial order \( \leq \)

For \( x, y \in X \) we say \( y \) covers \( x \) whenever

\( x < y \) and there does not exist \( z \in X \) such that

\( x < z < y \)

In this case write

\( x \prec y \)
Ex  Take \( X = \{1, 2, 3, \ldots, 10\} \)

Define a partial order \( \leq \) on \( X \) by

\[ x \leq y \text{ whenever } x \mid y \ (x \text{ divides } y) \]

Describe the carre relation \( <_c \)

Sol. For \( x, y \in X \) draw

\[
\begin{array}{c}
\text{Hasse diagram}
\end{array}
\]

where \( x <_c y \)
Given a set $S$, let $\mathcal{P}(S)$ denote the set of all subsets of $S$.

$\exists |\mathcal{P}(S)| = 2^n \quad n = |S|$

Observe the containment relation $\subseteq$ is a partial order on $\mathcal{P}(S)$.

Ex. Take $S = \{1, 2, 3\}$

$X = \mathcal{P}(S)$

Describe the Hasse diagram for the poset $X, \subseteq$

It's like a 3-cube!
Def: Given a set $X$ 

Given 2 partial orders on $X$: 

$\leq_1, \leq_2$ 

$\leq_2$ is called an extension of $\leq_1$ whenever 

$x \leq_1 y \implies x \leq_2 y \quad \forall x, y \in X$ 

Def: Given a partial order $\leq$ on a set $X$ 

A linear extension of $\leq$ is an extension of $\leq$ that is a total order. 

Then let $\leq$ denote a partial order on a finite set $X$, then $\leq$ has at least one linear extension. 

pf: Let $n = |X|$. 

Let $x_1$ denote a minimal element of $X$ 

Let $x_2$ . . . 

Let $x_3$ . . . 

. . .

This gives ordering $x_1, x_2, \ldots, x_n$ of elements of $X$
Define a total order on $X$ as follows.

In the total order

$x_i$ covers $x_j$ if $i - 1 < j$.

So, the Hasse diagram is

```
      x_1
       |
     x_2  |
       |
     x_3
       |
     x_4
```

One checks the total order is a linear extension of $\leq$.

**Ex.** Consider poset with Hasse diagram

```
    c
   /\  
  /   \ 
 b     d
   \   /  
    \ b  
```

Find all the linear extensions.

**Sol.**

\[ a < b < c < d < e \]

\[ a < b < d < c < e \]

\[ a < d < b < c < e \]
We now discuss equivalence relations.

Given set $\mathbf{X}$.

A relation $R$ on $\mathbf{X}$ is an equivalence relation whenever $R$ is

reflexive, symmetric, transitive.

Example: $S = \{1, 2, \ldots, n\}$

$\mathbf{X} = \mathcal{P}(S)$

Define a relation $R$ on $\mathbf{X}$ by

$xRy$ whenever $|x| = |y|$

then $R$ is an equivalence relation.
Ex  Given a set $X$

Partition $X$ into nonempty subsets

$$X = X_1 \cup X_2 \cup \cdots \cup X_r$$  (disjoint union)

Define a relation $R$ on $X$ as follows:

For $x, y \in X$

$$xRy \quad \text{where } x, y \text{ are in the same part of the partition}$$

Then $R$ is an equiv relation (check)

Ex  Given an equiv relation $R$ on a set $X$

For $x \in X$ define

$$[x] = \{ y \in X \mid xRy \}$$  \text{the equivalence class of } x

Observe $\forall x, y \in X$

$$[x] = [y] \quad \text{if } xRy$$

or

$$[x] \cap [y] = \emptyset \quad \text{if } x\notR y$$

Moreover the set of distinct equiv classes give a partition of $X$ into nonempty subsets.

---

A generic equivalence relation is usually denoted by $\sim$