Given $\Gamma = (X \times \mathbb{R})_{\text{diam } D}$

Assume $\{E_i\}_{i=0}^n$ is $Q$-poly ordering of the primal idempotents of $\Gamma$

Fix $x \in X$ write $T = T(x)$ etc.

Recall our goal is to prove Thm 56, 57.

**Notation 61**

(i) Given $\beta, \gamma, \delta \in \mathbb{F}$ define a 2-variable polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

(ii) Given $\beta, \gamma^*, \delta^* \in \mathbb{F}$ define a 2-variable poly

$$P^*(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma^*(\lambda + \mu) - \delta^*$$
LEM 62. Given $\beta, \gamma, \delta \in F$

$$0 = \left[ A, A^2 A^* - \beta A A^* A^* A^* - \gamma (A A^* A^* A) - \delta A^* A^* \right] \quad (\times)$$

iff

$$P(\alpha_i, \alpha_i) = 0 \quad \forall \alpha_i \neq 0$$

PF. Let $C = \text{RHS} \times (\times)$

$$C = \left( E_0 + E_1 + \cdots + E_n \right) C \left( E_0 + E_1 + \cdots + E_n \right)$$

$$= \sum_{i=0}^{0} \sum_{j=0}^{0} E_i C E_j$$

For $0 \leq i, j \leq 0$ use $E_i A = \delta_i E_i$ and $A E_j = \delta_j E_j$

to get

$$E_i C E_j = \left( \alpha_i - \delta_j \right) P(\alpha_i, \alpha_j) E_i A^* E_j$$

$\Rightarrow$: For $0 \leq i \leq 0$ show $P(\alpha_i, \alpha_i) = 0$

$$C = 0 \quad \text{so}$$

$$0 = E_{i-1} C E_i$$

$$= \left( \alpha_{i-1} - \delta_i \right) P(\alpha_{i-1}, \alpha_i) \frac{E_{i-1} A^* E_i}{\neq 0}$$
so \( o = P(e_i, e_i) \)

\[ \leq i \quad P \text{ is symmetric in its arguments so} \]

\[ P(e_i, e_{i+1}) = 0 \quad \text{for} \quad i \neq 0 \]

To show \( C = 0 \) show

\[ E_i C E_j = 0 \quad \text{for} \quad j = 1 \]

Given \( i, j \),

- If \( |i-j| > 1 \) then \( \lambda E_i A^* E_j = 0 \) so \( E_i C E_j = 0 \)
- If \( |i-j| = 1 \) then \( P(e_i, e_j) = 0 \) so \( E_i C E_j = 0 \)
- If \( i = j \) then \( E_i - E_j = 0 \) so \( E_i C E_j = 0 \)

In each case \( E_i C E_j = 0 \) so \( C = 0 \). \( \square \)
For the moment let \( \{ \alpha_{i,0} \} \) be any sequence of scalars in \( F \). Given \( \beta \in F \), call the sequence \( \beta \)-recurrent whenever

\[
\theta_{i-2} - (\beta\theta_i)\theta_{i-1} + (\beta\theta_i)\theta_i - \theta_{i+1} = 0
\]

\( M \geq i = 0 \ldots \). Given \( \beta, \gamma \in F \) call \( \{ \alpha_{i,0} \} \) \( (\beta, \gamma) \)-recurrent whenever

\[
\theta_{i+1} - \beta\theta_i + \gamma\theta_i = 0
\]

\( M \geq i = 1 \ldots \). Observe \( \forall \beta \in F \) TFAE

\( i.i \) \( \{ \alpha_{i,0} \} \) is \( \beta \)-rec

\( i.i \) \( \exists \gamma \in F \) s.t. \( \{ \alpha_{i,0} \} \) is \( (\beta, \gamma) \)-rec
LEM 63 Given integer $0 \leq 0$ and a sequence of scalars

$\{ \theta_i ; i \geq 0 \}$ from $\mathbb{F}$, Given $\rho, \epsilon \in \mathbb{F}$

(1) Assume $\{ \theta_i ; i \geq 0 \}$ is $(\rho, \epsilon)$-rec Then $\exists \delta \in \mathbb{F}$ s.t.

$P(\theta_{i+1}, \theta_i) = 0 \quad i \in \mathbb{Z}$

(ii) Assume $\exists \delta \in \mathbb{F}$ s.t.

$P(\theta_{i+1}, \theta_i) = 0 \quad i \in \mathbb{Z}$

Further assume $\theta_{i+1} \neq \theta_{i+1}$ for $i \in \mathbb{Z}$, then $\{ \theta_i ; i \geq 0 \}

is (\rho, \epsilon)$-rec.

Proof define

$\rho_i = \theta_i^2 - \beta \theta_i \theta_{i+1} + \alpha \theta_i - \gamma (\theta_{i+1} + \theta_i) \quad i \in \mathbb{Z}$

and observe

$\rho_i - \rho_{i+1} = (\theta_{i+1} - \theta_i)(\theta_{i+1} - \beta \theta_i + \epsilon \theta_i - \gamma)$

Hence $i \in \mathbb{Z} \leq 0$. Result follows.
Proof of th 57

First assume $D \geq 3$

By LEM 60 (with $R = A^2$, $S = A$) \( \exists \ z \in M \) s.t.

\[
\]  

(\#)

Recall \( \{ A^j \}_{j=0}^\infty \) is a basis for \( M \) so \( \exists \ p \in F[x] \)

with degree \( \leq 0 \) s.t.

\[
z = p(A).
\]

Let \( d = \text{degree of } p \). We show \( d = 3 \).

First suppose \( d > 3 \). Multiply each term in (\#) on left by \( E_0^x \) and on right by \( E_0^x \). Evaluate using L58 to get

\[
0 = c (E_0^x - E_0^x) E_0^x A^3 E_0^x
\]

\[c = \text{leading coeff of } p\]

This is contradiction.

Next suppose \( d < 3 \). Multiply each term in (\#) on the left by \( E_3^x \) and on the right by \( E_0^x \). Evaluate using L58 to get

\[
(E_3^x - E_0^x) E_3^x A^3 E_0^x = 0
\]

\[\# \quad \# \quad \# \quad \# \]

by L58

\text{cont.}
We have shown \( d = 3 \), abbrev \( \beta = \frac{\pi}{2} - 1 \).

Now divide both sides of \((*)\) by \( c \) to find \( \exists \gamma, \delta, \epsilon \in \mathbb{F} \) s.t.

\[
- \gamma (A^2 A^* - A^* A^2) - \delta (AA^* - A^* A)
\]

Rearranging terms we get \( \text{TD1} \). To get \( \text{TD2} \) put \( \gamma = 2 \leq \epsilon = 0 \).

Multiply each term in \( \text{TD1} \) on left by \( E_i^{\epsilon} \) and on right by \( E_i^{\epsilon} \). Simpify using \( L59 \) to get

\[
0 = E_i^{\epsilon} A^3 E_i^{\epsilon} \left( \theta_i^{\epsilon} - (\beta \mu) \theta_i^{\epsilon} + (\beta \mu) \theta_i^{\epsilon} - \theta_i^{\epsilon} \right)
\]

must be 0

So \( \{ \theta_i^{\epsilon} \}_{i=0} \) is \( \beta \)-rec.

So \( \exists \gamma, \epsilon \in \mathbb{F} \) s.t. \( \{ \theta_i^{\epsilon} \}_{i=0} \) is \( (\beta, \gamma, \epsilon) \)-rec

So by \( L63 \) \( \exists \delta, \epsilon \in \mathbb{F} \) s.t.

\[
p^*(\theta_i^{\epsilon}, \theta_i^{\epsilon}) = 0 \quad 1 \leq i \leq 0
\]

Now \( \beta, \gamma, \delta, \epsilon \) s.t. \( \text{TD2} \) by \( L62^* \)

We are done for \( D^{23} \)
Now assume \( 0 < 3 \)

Let \( \beta \in \mathbb{F} \) (arbitrary)

If \( 0 = 2 \) define

\[
y = \theta_0 - \beta \theta_1 + \theta_2
\]

and if \( 0 \neq 1 \) let \( y \in \mathbb{F} \) (arb)

By construction \( \{ \theta_i^2 \} = 0 \) is \((\beta, x)\)-rec

So by L63 \((i)\) \( \exists \ s \in \mathbb{F} \) s.t.

\[
P(\theta_i, \theta_i) = 0 \quad \text{for} \quad 1 \leq i \leq n.
\]

Now \( \beta, y, s \) satisfy T01 by L62.

Interchanging \( A, A^* \) in above argument. \( \exists \ y^*, s^* \in \mathbb{F} \) s.t.

\[
\beta, y^*, s^* \text{ sat T02}
\]
Prop 64. Given \( \beta, r, r', s, s' \in \mathcal{F} \) that satisfy

(i) the expressions

\[
\frac{\theta_{i^2} - \theta_{i^3}}{\theta_{i^4} - \theta_{i^5}}, \quad \frac{\theta_{i^7} - \theta_{i^8}}{\theta_{i^9} - \theta_{i^10}}
\]

are both equal to \( \beta \) for \( z \in i^3 : 0 \rightarrow 7 \)

(ii) \( Y = \theta_{i^2} - \beta \theta_{i^3} + \theta_{i^4} \quad (12i^3 : 0 \rightarrow 1) \)

(iii) \( Y' = \theta_{i^2} - \beta \theta_{i^6} + \theta_{i^7} \quad (12i^3 : 0 \rightarrow 1) \)

(iv) \( S = \theta_{i^8} - \beta \theta_{i^7} \theta_{i^9} + \theta_{i^10}^2 - r (\theta_{i^7} + \theta_{i^9}) \quad (1 \leq i \leq 0) \)

(v) \( S' = \theta_{i^8} - \beta \theta_{i^8} \theta_{i^9} + \theta_{i^10}^2 - r' (\theta_{i^8} + \theta_{i^9}) \quad (1 \leq i \leq 0) \)

PF (iv) From T01 and L62

(v) From T02 and L62

(iii) By L63 and (iv) above

(iii) Sim to (iii)

(v) \( \theta_{i^8} \) is \((\beta, r)\)-rec by (ii) so \( \theta_{i^9} \) is \( \beta \)-rec

Sim \( \theta_{i^9} \) is \( \beta \)-rec. Result follows
Prop 65: the scalars $p$, $r$, $x$, $s$, $s^*$ in $A57$

are unique provided $p 
eq 3$.

Proof: By Prop 64.

Proof of $A56$: Immediate from Prop 64 (c).
Ex 66 \[ T = H (0, N) \]

Recall \( T \) has a Q-poly structure such that
\[ \theta_i = \theta_i^* = (n - X (0 - i)) - i \quad (0 \leq i \leq n) \]

One checks that for this structure, the parameters
\[ \beta, \gamma, \delta, \delta^* \] from Theorem 5.4 are
\[ \beta = 2, \quad \gamma = 0, \quad \delta^* = N^2. \]
\[ \delta = N^2. \]

Then, \( \alpha_1, \alpha_2 \) become
\[ \left[ A, \left[ A, A^* \right] \right] = N^2 \left[ A, A^* \right], \]
\[ \left[ A^*, \left[ A^*, A \right] \right] = N^2 \left[ A^*, A \right]. \]

These equations are called the \( \alpha_1, \alpha_2 \) relations.
Note 67. The Onsager algebra $\mathfrak{O}$ is the Lie algebra over $F$ defined by generators $Y, Z$ and relations:

$$
\begin{align*}
\end{align*}
$$

where $[ , ]$ is Lie bracket. It turns out $\mathfrak{O}$ is 12-dimensional.

$\mathfrak{O}$ is used in the statistical mechanics of the Ising model.

By Ex. 66, we see that for $H(\mathcal{A}_N)$ the standard module is an $\mathfrak{O}$-module on which $Y, Z$ act as

$$
\frac{2A}{N}, \quad \frac{2A^*}{N}
$$

resp.
Note 68 \( \forall a, q \in \mathbb{F}, \quad q \neq 0, 1, -1 \)

Define the "\( q \) integer"

\[
\begin{align*}
\left[ n \right]_q &= \frac{q^n - q^{-n}}{q - q^{-1}} \\
&\quad n = 0, 1, 2, \ldots
\end{align*}
\]

The (cubic) \( q \)-Serre relations in the variables \( X, Y, Z \) are

\[
\begin{align*}
X^3 Y - \left[ 3 \right]_q X^2 Y X + \left[ 3 \right]_q X Y X^2 - Y X^3 &= 0, \\
Y^3 X - \left[ 3 \right]_q Y^2 X Y + \left[ 3 \right]_q Y X Y^2 - X Y^3 &= 0.
\end{align*}
\]

These are among the defining relations for the algebra \( U_q(\hat{sl}_2) \). This is the quantum group of the current matrix

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\]

The \( q \)-Serre relations are the same thing as the TDI, TDO relations with \( \beta = q^2 + q^{-2}, \quad \gamma = 0, \quad \gamma^* = 0, \quad \delta = 0, \quad \delta^* = 0 \).
\[
F = \mathbb{R} \times \mathbb{C} \quad \text{Given } \mathcal{O}_G \quad \Gamma = (x_i \mathbb{R}) \quad \text{dim } D
\]

Assume \( \{ E_i \}_{i=0}^p \) is \( \mathcal{O}_I \)-pol

Fix \( x_i \in \mathcal{X} \) and write \( T = T(x) \) etc.

We now solve the equations in Item 56 to get the eigenvalues and dual eigenvalues of \( \Gamma \) in closed form.

**LEM 69** Given a finite sequence \( \{ \theta_i \}_{i=0}^p \) of scalars in \( \mathcal{O} \), and given \( \beta \in \mathcal{O} \). Then \( \{ \theta_i \}_{i=0}^p \) is \( \beta \)-rec iff \( f, a, b, c \in \mathcal{O} \) such that

**Case** \( \beta = \pm 2 \)

\[ \theta_i = a_{i+1} + c_i \quad (0 \leq i \leq 0) \]

where \( f(i) = \beta \)

**Case** \( \beta = 2 \)

\[ \theta_i = a_{i+1} + c_i \quad (0 \leq i \leq 0) \]

**Case** \( \beta = -2 \)

\[ \theta_i = a_{i+1} + c_i(-1) \quad (0 \leq i \leq 0) \]
Proof (For case $p \neq 2$)

Assume $p \geq 3$ else trivial

Let $L$ denote the set of all vectors $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ in $\mathbb{C}^{n+1}$

but are $\beta$-rec, i.e.

$$\sigma_{i+2} - (\beta+1) \sigma_{i+1} + (\beta+2) \sigma_i - \sigma_i = 0 \quad (2 \leq i \leq n-1) \quad (*)$$

Obs $L$ is a subspace of $\mathbb{C}^{n+1}$

In $(*)$, $\sigma_0, \sigma_1, \sigma_2$ are free and $\sigma_3, \ldots, \sigma_n$ are determined by $\sigma_0, \sigma_1, \sigma_2$

so $\dim L = 3$

Pick $q \in \mathbb{C}$ s.t.

$$\beta = q + q^{-1}$$

Obs $q \neq 1, q \neq -1$

One checks the three vectors

$$(1, 1, 1, \ldots, 1), \quad (1, q^2, q^4, \ldots, q^n), \quad (1, q^3, q^5, \ldots, q^{2n})$$

are in $L$ and linearly independent. So they form a basis for $L$. Result follows. □
Note 70  Ref to L69, for $\beta \neq \pm 2$ sometimes we replace $q, b, q^2$ and adjust $b, c$ to write

$$\theta^i = a + b q^{2i-1} + c q^{0-2i} \quad (0 \leq i \leq 0)$$

Cor 71  Referring to $\Gamma$, assume $0 \geq 3$ to avoid miracles.

Let $\beta \in F$ be from $\Rightarrow 57$. Then the eigenvalues $\{\theta^i\}_{i=0}^0$ of $\Gamma$ and dual eigenvalues $\{\bar{\theta}^i\}_{i=0}^0$ of $\bar{\Gamma}$ satisfy one of the following forms.

**Case I:** $\beta \neq \pm 2$

$$\theta^i = a + b q^{2i-1} + c q^{0-2i}$$

$$\bar{\theta}^i = a^* + b^* q^{2i-1} + c^* q^{0-2i} \quad 0 \leq i \leq 0$$

$$\beta = q^2 + q^{-2}$$

**Case II:** $\beta = 2$

$$\theta^i = a + b i + c i^2 \quad 0 \leq i \leq 0$$

$$\bar{\theta}^i = a^* + b^* i + c^* i^2$$

**Case III** $\beta = -2$

$$\theta^i = a + b (1i) + c i (-1) i$$

$$\bar{\theta}^i = a^* + b^* (-1) i + c^* i (-1) i \quad 0 \leq i \leq 0$$
Caution Ref to Cor 71, possibly some of

$q, \alpha, i, \omega, i^*, c^* \text{ are in } \mathbb{C} \setminus \mathbb{R}

even though the $\alpha, i, c^*$ are all in $\mathbb{R}$
Note 72. Ref to Cor 71. for Case I.

The parameters \( \beta, \eta, \nu, \delta, \lambda \) from that are:

\[
\beta = q^2 + q^{-2}
\]

\[
\nu = -a(q^n q^{-n})^2
\]

\[
\nu^* = -a^*(q^n q^{-n})^2
\]

\[
\delta = -6c(q^n q^{-n})^2 + a^2(q^n q^{-n})^2
\]

\[
\delta^* = -6c^*(q^n q^{-n})^2 + a^{*2}(q^n q^{-n})^2
\]

This is checked using Prop 64. Similar equations hold for cases II, III.
Note 73  Earlier we found a Q-pol3 structure/f

\[ \Gamma = H(Q,N). \quad \text{It was Case II with} \]

\[ a = a^* = (N-1)0 \]

\[ b = b^* = -N \]

\[ c = c^* = 0 \]
Obs our ORG $\Gamma$ is bipartite iff $a_i=0 \forall 0 \leq i \leq D$

Call our $Q$-poly structure dual bipartite whenever

$$a_i^* = 0 \quad \forall 0 \leq i \leq D.$$ 

Th 74. Let the scalars $\beta, \gamma, \delta, \delta^*$ be in $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$. Assume $D=2$

(i) Assume $\Gamma$ is bipartite, then

$$0 = A^2 A - \beta A^* A A^* + A A^* \gamma^* (AA^* A A^*) - \delta^* A \tag{(*)}$$

and $\gamma = 0$

(ii) Assume the Q-poly str is dual bipartite, then

$$0 = A^2 A^* - \beta A A^* A + A^* A^* \gamma - \delta (AA^* A A^*) - \delta A^*$$

and $\gamma^* = 0$

pf (i) Let $F = RHS \tag{(*)}$ Show $F=0$ Obs

$$F = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* F E_j^*$$

For $0 \leq i \leq D$ show $E_i^* F E_i^* = 0$. Obs

$$E_i^* F E_j^* = P^*(\Theta_i, \Theta_j) E_i^* A E_j^*$$

where $P^*$ is from Not 61

Case 1 $|\gamma| > 1: \quad E_i^* A E_i^* = 0$
Case $i=1$: \( P^* (e_1^*, e_1^*) = 0 \) since the dual eigenvalues are \( p \)-rec

Case $i=2$: \( E_i^* A E_i^* = 0 \) since \( q_i = 0 \)

In all cases \( E_i^* F E_i^* = 0 \)

so \( F = 0 \).

Show \( x = 0 \)

Recall back in Lec 8 of Ch 1, we found

\[ H \in \text{Mat}_{N \times N}(F) \text{ s.t.} \]

\[ HA = -AH \]

Replacing \( H \) by \( -H \), hence

\[ H = \sum_{i=0}^{d} (-1)^i E_i^* \]

Obs \( H^* A H = H^2 = I \). By that we have \( 701\% \)

\[ 0 = \begin{bmatrix} A, A^2 A^* - \beta A^* A^* + A^* A^2 - \delta (A^* A^* - \gamma I) - \delta A^* \end{bmatrix} \]
Conjugate $R^0$ by $H$ to get

$$0 = \left[ -A^0, (-A^0)^2 - p(-A^0)A^0(-A^0) + A^0(-A^0)^2 - \delta (A^0 A^0 - A^0 A^0) - \delta A^0 \right]$$

So

$$0 = \left[ A^0, A^0 A^0 - \beta A^0 A^0 + A^0 A^0 + \delta (A^0 A^0 - A^0 A^0) - \delta A^0 \right]$$

Subtracting $0$ from $TDI$.

$$0 = \delta \left[ A^0, A^0 A^0 + A^0 A^0 \right]$$

$$= \delta \left[ A^0, A^0 \right]$$

But $\left[ A^0, A^0 \right] \neq \text{some}$

$E^0_0 \left[ A^0, A^0 \right] E^0_0 = E^0_0 A^0 \stackrel{0}{E} \left( E^0_0 - \delta \right) \frac{E^0_0}{\delta}$

So $\delta = 0$

$$\text{c.c. Sim.}$$
Ex 75 Take $\Gamma = H(0,2)$ hyperbolic. Take $F = \mathbb{C}$

$\Gamma$ is bipartite, the only structure we found earlier satisfies

$q_i^b = p_i^b$ for $0 \leq b, i \leq 0$

So this structure is dual bipartite.

Recall

$q_i^* = p_i^* = 0 - 2i$ \hspace{1cm} (or i = 0)

Here $\beta = 2$

$r = 0, \quad r^* = 0 \hspace{1cm} \delta = 4, \quad \delta^* = 4$

So by Th 74

$4A = A^*A - 2A^*AA^* + AA^*^2$

$= \left[ A^*, \left[ A^*, A \right] \right]$ \hspace{1cm} (*)


$= \left[ A, \left[ A, A^* \right] \right]$ \hspace{1cm} (**)

For notational convenience define $A^\varepsilon$ by

$\left[ A, A^* \right] = 2iA^\varepsilon \hspace{1cm} (i^2 = -1)$
Then (x), (x*) become

\[
\begin{align*}
\left[ A^x, A^e \right] &= z_0 A \\
\left[ A^e, A^x \right] &= z_i A^*
\end{align*}
\]

We now recognize the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \)

Recall \( \mathfrak{sl}_2(\mathbb{C}) \) is the Lie algebra consisting of all \( 2 \times 2 \) matrices over \( \mathbb{C} \) with trace 0. The Lie bracket is

\[
\left[ x \cdot y \right] = x^y - y^x, \quad \forall x, y \in \mathfrak{sl}_2(\mathbb{C})
\]

\( \mathfrak{sl}_2(\mathbb{C}) \) has a basis:

\[
\begin{align*}
a &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & a^* &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & a^e &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\left[ a, a^e \right] &= z_0 a^e, & \left[ a^*, a^e \right] &= z_0 a, & \left[ a^e, a^* \right] &= z_i a^*
\end{align*}
\]

Thus for \( H(0, 2) \) the standard module becomes \( \omega \)

\( \mathfrak{sl}_2(\mathbb{C}) \)-module at \( a, a^*, a^e \) act as \( A, A^*, A^e \) resp.
Given DRG $\Gamma = (X; R)$ diam $D \geq 1$

Assume ordering $\{ i; j; k = 0 \}$ is $Q$-poly.

Fix $x \in X$ write $T = T(x)$ etc.

We mention a handy formula.

**LEM 76** With above notation

\[(i) \quad c_i \theta^i + a_i \theta^i + b_i \theta^i = \theta^i\]

where $\theta^i$ are undets.

\[(ii) \quad c_i \theta^i + a_i \theta^i + b_i \theta^i = \theta^i\]

where $\theta^i$ are undets.

**Proof** (i) By AV duality

\[u_i(\theta) = u_i^*(\theta)\]

\[= \frac{\theta^i}{\theta_0^i}\]

Result follows from this and the 3-term vec for $u_i^*$. 

(ii) Sim
Setting $i=0$ in LEM 76 (i) and using $k=\theta_0$

\[ \frac{\theta_i}{\theta_0} = \frac{\theta_i}{\theta_0} \]

**LEM 77.** Assume $\Gamma$ is Bipartite

(i) \[ c_i = \frac{\theta_0}{\theta_i} = \frac{\theta_1\theta_i - \theta_0}{\theta_1 - \theta_0} \]

(ii) \[ c_0 = \theta_0 \]

(iii) \[ b_i = \theta_0 - c_i \quad (\theta \in \theta_0) \]

(iv) \[ \frac{\theta_0 - c_i}{\theta_0} = \frac{\theta_i}{\theta_0} \]

**Pf (i)** In LEM 76 (i) Eval using $\alpha_i = 0$, $b_i = k - c_i$, and solve for $c_i$

(iii) \[ c_0 = k - \theta_0 = k = \theta_0 \]

(iv) \[ k = \theta_0 \]

(v) Set $i=0$ in LEM 76 (i)
LEM 78  \[ \text{Assume } \sum_{i=3}^{\infty} e_i^2 \leq \epsilon \text{ is dual bipartite.} \]

(i) \[ c_i^* = \frac{\theta_0^*}{\theta_0} \quad \frac{\theta_1 \theta_i - \theta_0 \theta_i^*}{\theta_1 - \theta_i^*} \quad 1 \leq i \leq 0 \]

(ii) \[ c_0^* = \theta_0^* \]

(iii) \[ v_i^* = \theta_0^* - c_i^* \quad 0 \leq i \leq 0 \]

(iv) \[ \frac{\theta_{0-1}}{\theta_0} = \frac{\theta_1}{\theta_0} \]

PF  Sim to LTT
LEM 79  Assume $R$ is bip and $\{E: \exists :\}$ is dual bip.

Further assume $\beta = 2$. Then

(i) $\theta_0 = \theta_0^* = 0 - 2i$ \hspace{1cm} (0 < \epsilon < \theta_0)

(ii) $\theta_i = \theta_i^* = i$ \hspace{1cm} (0 < \epsilon < \theta_0)

"It looks like $H(0, 2)$"

pf  Assume $D \geq 2$ else trivial.

We are in Case II; the $\theta_i, \theta_i^*$ have form

$\theta_i = a + bi + c_i i^2$ \hspace{1cm} 0 \leq \epsilon < \theta_0

$\theta_i^* = a + bi + c_i^* i^2$

By $H 74$ (i)

$0 = \gamma$

$\gamma = \theta_0 - \beta \theta_i + \theta_2$

$= a - 2(a \nu b + c) + a + 2i + 4c$

$= 2c$

So  $c = 0$
Sim \[ c^* = 0 \]

the constraint

\[
\frac{\theta_{0+1}}{\theta_0} = \frac{\theta_{1}}{\theta_0}
\]

from LTT (iv) gives

\[
b/a = -\frac{2}{a}
\]

Similarly using LTT (iv)

\[
b/\theta^* = -\frac{2}{\theta}
\]

So far

\[
\frac{\theta_{c}}{\theta_0} = 1 - \frac{2c}{\theta}
\]

\[0 \leq c \leq 0\]

\[
\frac{\theta_{c}}{\theta_0} = 1 - \frac{2c}{\theta}
\]

\[0 \leq c \leq 0\]

For \(1 \leq c \leq 1\) solve for \(c_i\) using LTT (iv) to get

\[c_i = \frac{c_0}{a}
\]

But \(c_i = 1\) so

\[\theta_0 = 0
\]

hence

\[\theta_i = b - 2c_i\]

(0 \leq 2a)
Also

\[ c_0 = \theta_0 = 0 \]

\[ c_i = \theta_i \quad (i \leq i_{20} - 1) \]

\[ \epsilon_0 \]

\[ c_i = \theta_i \quad (0 \leq i_{20}) \]

Similarly

\[ \theta_i^* = 0 - 2i \quad (0 \leq i_{20}) \]

\[ c_i^* = i \quad (0 \leq i_{20}) \]
Ex TFA E

(:1) \( \Gamma \) is bipartite and

\[ \epsilon_i = \epsilon \quad (0 \leq i \leq \ell) \]

(:3i) \( \Gamma \) is \( H(0,2) \)

Proof (3a) Consider set of vectors

\[ \left\{ E_i \hat{y} - E_i \hat{z} \mid y, z \in X, \ y \neq z \in R \right\} \]

Show for all \( u, v \) in this set either

\[ u = \bar{v} \quad \land \quad \langle u, v \rangle = 0 \]

Use this to show \( \Gamma = K_2 \times K_2 \times \cdots \times K_2 \).

0
LEM 80. Assume $\Gamma$ is bip and $\delta \in \delta_{\text{iso}}$ is dual bip. Further assume $\beta \neq \pm 2$.

Then $\forall q \in \mathbb{C}$ $(q^2 \neq 1, q^2 \neq -1)$ s.t.

\[(i) \quad \theta_i = \theta_i^\phi = \left( q^{0-2} + q^{2-0} \right) \frac{q^{2i-2} - q^{2i-0}}{q^2 - q^{-2}} \quad (0 \in \mathbb{Z}) \]

\[(ii) \quad \zeta_i = \zeta_i^\phi = \frac{q^{0-2} + q^{2-0}}{q^{2i-2} + q^{2i-0}} \frac{q^{2i-2} - q^{2i-0}}{q^2 - q^{-2}} \quad (0 \in \mathbb{Z}) \]

\[\text{Pf.} \quad \text{Sim to pt. of L79, except one Case IV terms for } \theta_i \phi^\delta. \]
LEM 8: Assume $E$ is bip and $E^{\circ}$ is dual bip

Further assume $\beta = -2$. Then $D$ is even and

(i) $\sigma_i = \sigma_i^* = (-1)^i (0-2\alpha) \quad (0 \leq i \leq 0)$

(ii) $c_i = c_i^* = \delta \quad (0 \leq i \leq 0)$

PF: Sim to pf 4 L79 except one case III from $\sigma_i \sigma_i^*$.
Note: Unique sol to $\mathbb{E}(\pi, 2)$ is $H(\pi, 2)$ (0 even)

with $Q$-poly shr assoc with ordering

$D, 2\pi, \theta \pi, \xi \pi, \ldots$

of the eigenvalues.

(Fun fact: this really is a $Q$-poly shr.)
With ref to L80

\[ \beta = q^2 + q^{-2} \]

\[ r = r^* = 0 \]

\[ s = s^* = \left( q^{p-2} + q^{2-z} \right)^2 \]

By L74

\[ A^x A^* - (q^2 + q^{-2}) A A^* A + A^x A^2 = \left( q^{p-2} + q^{2-z} \right)^2 A \]

\[ A^* A = (q^2 - q^{2-z}) A A^* A + A A^* = \left( q^{p-2} + q^{2-z} \right)^2 A \]

In cyclic form this looks as follows.

LEM 82 With ref to L80 \( \exists A^x \in T \) s.t.

\[ q A^x A - q^* A^* A = z A^E \]

\[ q A^* A^E - q^* A^E A^* = z A \]

\[ q A^E A - q^* A A^E = z A^* \]

where

\[ z = \frac{1}{2} (q^{p-2} + q^{2-z}) \]

\[ z^2 = -1 \]

pf Routine - just def \( A^E \) using 1st equation.
\[ F = C \quad \text{Given bipartite or } G \quad \Gamma = (X, \mathbb{R}) \]

draw \( D \geq 2 \)

Assume \( \{ E_i \}_{i=0}^{p} \) is dual bip \( \Phi \)-only

Further assume \( \beta \neq \pm 2 \), let \( g \in C \) be a in \( L^2 \)

Fix \( x \in X \) into \( T = T(x) \) etc.

Last time we found \( A^\varepsilon \in T \) s.t.

\[
q A A^\varepsilon - q^* A^\varepsilon A = z A^\varepsilon,
\]

\[
q A^\varepsilon A^\varepsilon - q^* A^\varepsilon A^\varepsilon = z A,
\]

\[
q A^\varepsilon A^\varepsilon - q^* A A^\varepsilon = z A^\varepsilon,
\]

where \( z = \varepsilon \left( q^{0-2} + q^{2-2} \right) \) \( \varepsilon^2 = -1 \)

Next goal: show \( A^\varepsilon \) is an imaginary adj matrix

Search for \( W \in M \) and \( W^* \in M^* \)

set

\[ WA^\varepsilon W^* = W^{*\varepsilon} A W^* = A^\varepsilon \]

(just like we did for \( F \))
Find \( W \): For moment assume \( W \) exists. Write

\[
W = \sum_{i=0}^{0} \alpha_i E_i, \quad \alpha_i \in \mathbb{C}
\]

\( W \) exists so \( \alpha_i \to 0 \) (or \( \alpha_i \to n \)) and

\[
W^* = \sum_{i=0}^{0} \alpha^*_i E_i.
\]

Require

\[
A^E = W A^* W^*
\]

For \( \theta \neq i, j \neq 0 \)

\[
E_i A^E E_j = E_i \left( W A^* W^* \right) E_j
\]

\[
= E_i A^* E_j \quad \alpha_i / \alpha_j
\]

Also

\[
E_i A^E E_j = E_i \left( \frac{q A A^* - \gamma A^* A}{\gamma} \right) E_j
\]

\[
= E_i A^* E_j \quad \frac{\gamma \Theta_i - \gamma \Theta_j}{\gamma^2}
\]

Since we assume our Q-polystr is dual bip

\[
E_i A^* E_j = 0 \quad \text{if} \quad (i \neq j) \neq 1
\]
\[ \frac{d^2}{d^2} = \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \text{if} \quad |i-\text{ref}| = 1 \quad (0 \leq i \leq I) \]

Recall \( \delta \leq 0 \)

\[ \delta = \frac{\Theta_1}{\Theta_1} - \frac{\beta \Theta_1}{\Theta_1} \cdot \Theta_1 + \Theta_1^2 \quad (\theta = \Theta_1 + \Theta_2) \]

\[ -\delta = \left( q \Theta_1 + q^* \Theta_1 \right) \left( q \Theta_1 - q^* \Theta_1 \right) \]

\[ 1 = \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \text{if} \quad |i-\text{ref}| = 1 \quad (0 \leq i \leq I) \]

So \( \delta \leq 0 \)

\[ 1 = \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \text{if} \quad |i-\text{ref}| = 1 \]

Only requirement on \( \{ x_i \} \leq \) is

\[ \frac{d^2 x_i}{dx_i} = \frac{q \Theta_1 - q^* \Theta_1}{z} \quad \leq \leq \]

By L20

\[ \frac{q \Theta_1 - q^* \Theta_1}{z} = \frac{z^2 \Theta_1 - 1}{e} \quad \leq \leq \]

\( (x^2 = r) \)
Def 83 With above notation put \( 0 + x_0 \in \mathbb{C} \)

and define \( \{ x_i \}_{i=0}^p \) by

\[
\frac{x_i}{x_{i-1}} = x_0^{2^{i-1} - 1} \quad (1 \leq i \leq p)
\]

Put

\[
W = \sum_{i=0}^{p} x_i E_i
\]

\[
W^* = \sum_{i=0}^{p} x_i E_i^*
\]
Prop 84

With above not

(i) \(WA^*W^* = A^E\)

(ii) \(W^{*r}AW^* = A^E\)

pf (i)

\[
A^E = \left( E_0 + E_1 + \cdots + E_\infty \right) A^E \left( E_0 + E_1 + \cdots + E_\infty \right)
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} E_i A^E E_j
\]

Also,

\[
WA^*W^* = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} E_i (WA^*W^*) E_j
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_i \alpha_j E_i A^* E_j
\]

Fm. \(0 \leq i, j \leq n\)

\[
E_i A^E E_j - \alpha_i \alpha_j E_i A^* E_j
\]

\[
= \frac{E_i A^* E_j}{\frac{\gamma E_i - \gamma E_j}{2}} - \frac{\alpha_i}{\alpha_j}
\]

\[
= 0 \quad \text{pf (ii) Very sim}
\]
COR 85  With above not

$A^E$ is similar to $A$, $A^v$

In particular $A^E$ is diagonalizable with distinct eigenvalues

$\theta_i^E = \theta_i = \theta_i^v$  \(0 \leq i \leq D\)
Prop 86 With above not

\[ \begin{array}{c}
A^* \\
\rightarrow \\
A^* \\
\end{array} \]

where

\[ T \rightarrow T \]

\[ \rho : \]

\[ m \rightarrow (ww^*)m(ww^*)^\sim \]

Pf (just like LEM51)

check \( A \rightarrow A^* \):

\[ ww^*A = A^*ww^* \]

By crisis

\[ wA^*w^* = \left( w^\sim \right)Aw^* \]

so

\[ w^*wA^* = Aw^*w \]

take trans

\[ ww^*A = A^*ww^* \]
check $A^v \rightarrow A^e$:

$$w w^* A^v \ ? = A^e w w^*$$

$$A^e w w^* = w A^v w^* w w^*$$

$$= w A^* w^*$$

$$= w w^* A^* \checkmark$$

check $A^e \rightarrow A$

$$w w^* A^e \ ? = A w w^*$$

$$w w^* A^e = w w^* (w^* A w^*)$$

$$= w A w^*$$

$$= A w w^* \checkmark$$

Then 87 with above not,

$A^e$ is an imaginary adj matrix for $\Gamma$ in sense of Def 55.

Pf. By Prop 86 any two of $A, A^v, A^e$ related the same way as $A, A^v$. \qed
Problem 88  In our study of $KN$ we obtained

many results concerning $A, A^*, A^E, W, W^*.$

Try to find similar results for the Bip/dual Bip case.

$\left[ \text{is } W \text{ a } q\text{-exponential in } A? \quad \text{See if } W \text{ is an} \right.$

exponential in $A.$ $f \in H(\mathbb{R}_+). \quad \text{Repeat for Case III} \right]$
\[ F = R \otimes V \quad \Gamma = (x \otimes 1) \quad \text{any } D \otimes G \quad \text{where } D \geq 2 \]

Assume \( \{ E : 1 \to G \} \) is \( G \)-poly

Fix \( x \in X \), write \( \Gamma = \Gamma(x) \) etc.

Fix an \( \Gamma \)-module \( W \).

Next goal is to carefully study \( W \).

\[ r = \text{cosh of } W \quad \quad \quad \quad \quad t = \text{dual cospht of } W \]
\[ d = \text{diam } W \quad \quad \quad \quad \quad d^* = \text{dual diam of } W \]

**LEM 89** With above not

(i) \( E_i \otimes A \otimes V \otimes W = 0 \) if \( l - l' = 1 \) \( (r \leq i \leq r + d) \)

(ii) Suppose \( W \) is then. Then

\[ E_r W + E_{r+1} W + \cdots + E_m W = E_r W + A E_r W + \cdots + A^s E_r W \]
\[ (0 \leq s \leq d) \]

(iii) Suppose \( W \) is then. Then \( W = \mathfrak{M}_{E_r} W \)

(iv) Suppose \( W \) is then. Then

\[ E_l E_r W = E_l W \quad (0 \leq l \leq D) \]

Moreover, \( W \) is dual then.
\[ \text{Pf (i): Suppose } \exists \ i, j \ (s \leq i, j \leq r + k) \ s.t. \]

\[ E_i \mathbf{A} E_j^* W = 0 \quad \text{and} \quad \|i-j\| = 1 \]

If \( i - j = 1 \) then

\[ E_i^* W + \cdots + E_j^* W \]

is a non-zero proper subspace of \( W \) that is inv under \( A \mathbf{A}^* \), and \( \text{med} W \).

If \( j - i = 1 \) then

\[ E_j^* W + \cdots + E_i^* W \]

is a non-zero proper subspace of \( W \) that is inv under \( A \mathbf{A}^* \), and \( \text{med} W \).

(iii) By (i) and since

\[ E_i^* A E_j^* = 0 \quad \text{if} \quad |i-j| > 0 \quad (0 \leq i, j \leq 0) \]

(iii) \( \exists i \) Since \( W \) is \( M \)-inv

Set \( i = d \) in (iii) and use

\[ W = \sum_{i=0}^{d} E_i^* W \]
(iv) \( F_n \neq \emptyset \Rightarrow 0 \)  

\[ E_j W = E_j M E_r^* W \]

\[ = E_j E_r^* W \quad E_j M = \text{span} (E_j) \]

\[ \forall \omega \quad \dim E_j W \leq \dim E_r^* W \]

\[ = 1 \]

\[ \square \]
L 90 With alone not

(i) \( E : A^* E, W \rightarrow 0 \) \( i = k + 1 \) (this is true)

(ii) Suppose \( W \) is dual then then

\[ E \cdot W + E \cdot W = E \cdot W + A \cdot E \cdot W = 0 \text{ (since \( d \))} \]

(iii) Suppose \( W \) is dual then then \( W = M \cdot E \cdot W \)

(iv) Suppose \( W \) is dual then then

\[ E^* A \cdot E \cdot W = E^* W \]

Main: \( W \) is dual.
$F = \mathbb{R} \times \mathbb{C}$

Given $D\mathbb{R}^n \times (x \in \mathbb{N}) \text{ diam } \geq 2$

Assume $\{E_i^0\} \# \text{ is } \emptyset \text{ poly}$

Fix $x \in X$, write $T = T(x)$ etc.

Goal: study $\text{str of single model } T$-module

LEM 91 The following hold for $0 \leq h \leq 0$

(i) $F_0 + \omega = E_h^0$ $V$

$$\left| \left\{ x' \mid 0 < \varepsilon, E_i w = 0 \right\} \right| \leq 2h$$

(ii) $F_0 + \omega = E_h^0$ $V$

$$\left| \left\{ x \mid 0 < \varepsilon, E_i^0 w = 0 \right\} \right| \leq 2h$$

Pf (i) Suppose not. Then $\exists$ subset

$$\mathcal{S} \subseteq \{0, 1, \ldots, 0\} \quad |\mathcal{S}| = 2h + 1$$

s.t.

$$E_i w = 0 \quad \forall i \in \mathcal{S}$$

By construction $w = E_h^0$ $w$
So \( \forall \ i \in \Omega \)

\[
o = \sum_{i} E_{h}^{*} E_{i} E_{h}^{*} w = E_{h}^{*} \left( \sum_{i} E_{h}^{*} A_{i} \right) E_{h}^{*} w
\]

\[
E_{h}^{*} A_{i} E_{h}^{*} = 0 \text{ if } i > z h
\]

So

\[
o = \sum_{i=0}^{z h} u_{j}^{(i)} E_{h}^{*} A_{i} E_{h}^{*} w
\]

Letting \( j \) range over \( \Omega \) get system of \( z h + z \) homogeneous linear equations in the unknowns

\[
E_{h}^{*} A_{i} E_{h}^{*} w = 0 \quad 0 \leq i \leq z h
\]

The coefficient matrix is essentially Vandermonde since \( \text{poly} \ u_{j} \) has degree \( z \) and the \( A_{i} \) are distinct.

So the coefficient matrix is nonsingular hence

\[
E_{h}^{*} A_{i} E_{h}^{*} w = 0 \quad 0 \leq i \leq z h
\]

This is impossible since \( E_{h}^{*} A_{0} E_{h}^{*} w = w \neq 0 \)

\( \Box \)
COR 92 Let $W$ denote an ideal $F$-module with.

Consider dual $r$, dual subset $s$, given $d$, dual form $d^*$. Then

1. $2r + d^* \geq 0$
2. $2s + d \geq 0$

**Proof** (i) Fix $0 \neq w \in E_r^* W$. By L91 (i)

$$2r \geq \left| \left\{ i \right\mid 0 \leq i \leq 0, E_i w = 0 \right\} \right|$$

$$\geq \left| \left\{ i \right\mid 0 \leq i \leq 0, E_i W = 0 \right\} \right|$$

$$= 0 - d^*$$

by def of $d^*$

(ii) Sim
Until further notice let \( W \) denote a thin unred T-module with indept \( r \), dual indept \( d \),

dean \( d = 0 \) or dual dean \( d^* = d \)

**LEM 93** With above notation

(i) For all \( o \neq \eta \in E_r W \) the vector

\( E_{\eta^*} \) is a basis for \( E^*_r W \) for \( r \neq s \) or \( r = s \) and \( \eta \neq \eta^* \).

Moreover

\( E_{\eta}, E_{\eta^*}, \ldots, E_{\eta^*} \)

is a basis for \( W \)

(\xi) For all \( o \neq \eta \in E_r^* W \) the vector

\( E_{\eta^*} \) is basis for \( E^*_r W \) for \( r \neq s \) or \( r = s \) and \( \eta \neq \eta^* \).

Moreover

\( E_{\eta^*}, E_{\eta^*}, \ldots, E_{\eta^*} \)

is a basis for \( W \)
pf (i) By constr. for $0 \leq i \leq 0$

$E_i^*W$ has dim 1 if $r \leq i \leq rd$, and $E_i^*W = 0$

otherwise.

So for $r \leq i \leq rd$ set to show $E_i^*W = 0$.

By (iv)

$E_i^*W = E_i^*E_0W$

$= \text{Span } E_i^*$

$E_0W = \text{Span } \{z\}$

Result follows.

Cite Sim. \qed
DEF 94 With above notation

(i) By a standard basis for $W$ we mean a sequence

$$E_r^1, E_r^2, \ldots, E_r^n$$

where $0 + r \in E_r^1 W$

(ii) By a dual standard basis for $W$ we mean a sequence

$$E_r^1, E_r^2, \ldots, E_r^n$$

where $0 + r \in E_r^1 W$. 

We mention how to recognize a standard basis.

**LEM 95** Let \( \{w_i\}_{i=0}^d \) denote a sequence of vectors in \( W \), not all 0.

Then \( \{w_i\}_{i=0}^d \) is a standard basis for \( W \) iff both

\[
(i) \quad w_i \in \mathbb{E}^\perp_{w_{i-1}} W \quad \text{and} \quad w_i \neq 0.
\]

\[
(ii) \quad \sum_{i=0}^d w_i \in E_{L^W} W.
\]

**Pf** \( \Rightarrow \) \( \text{clear} \)

Define \( \gamma = \sum_{i=0}^d w_i \).

So \( \gamma \in E_{L^W} W \).

By construction

\[
w_i = \mathbb{E}^\perp_{w_{i-1}} \gamma \quad \text{and} \quad w_i \neq 0.
\]

Our \( \gamma \) to the \( w_i \)'s \( \text{shows} \quad \square \)

A similar result holds for dual standard bases.
Notation  

for an integer \( d \geq 0 \)

\[ \text{Mat}_d(\mathbb{F}) \]

is the algebra of all \( d \times d \)

matrices over \( \mathbb{F} \).

Rows/Cols indexed by \( 0, 1, \ldots, d \).

Here is another way to recognize the basis:

**LEM 9.6**

Let \( \{ w_i \}_{i=0}^d \) denote any basis for \( W \).

Let \( B \) (resp. \( B^* \)) denote the matrix in \( \text{Mat}_d(\mathbb{F}) \) that represents \( A \) (resp. \( A^* \)) w.r.t. \( \{ w_i \}_{i=0}^d \). Then \( \{ w_i \}_{i=0}^d \) is a basis for \( W \) if both

\[
\begin{align*}
(i) & \quad B \text{ has constant row sum } \alpha \\
(ii) & \quad B^* = \text{diag}(\alpha^*, \alpha^*, \ldots, \alpha^*)
\end{align*}
\]
Put

\[ y = \sum_{i=0}^{d} w_i \]

Obs

\[ A_y = \sum_{i=0}^{d} w_i \left( b_{i0} + b_{i1} + \cdots + b_{id} \right) \]

So

\[ \beta \text{ has const row sum at } \mathbf{y} \text{ for } y \in E \times W \]

Result follows. \[ \square \]
DEF 97

(i) We associate with \( W \) a \( F \)-alg hom
\[ b : T \rightarrow \text{Mat}_{n \times n}(F) \]
as follows. \( \forall y \in T, \ y^b \) is the matrix
rep \( y \) in a standard basis for \( W \).

(ii) We associate with \( W \) a \( F \)-alg hom
\[ \# : T \rightarrow \text{Mat}_{n \times n}(F) \]
as follows. \( \forall y \in T, \ y^\# \) is the matrix
rep \( y \) in a dual standard basis for \( W \).
LEM 98

(i) $A^b$ has constant row sum $\Theta e$

(ii) $A^*b = \text{diag} (e^*, e^*, \ldots, e^*)$

(iii) $A^# = \text{diag} (e_1, \ldots, e_n)$

(iv) $A^{*#}$ has constant row sum $e_{r^*}$

Pf: By L96 and dual \qed
DEF 99

(i) We define scalars

\[ c_i(w), a_i(w), b_i(w) \]

by

\[
A^b = \begin{pmatrix}
  a_0(w) & b_0(w) \\
  c_1(w) & a_1(w) & b_1(w) \\
  & \ddots & \ddots \\
  & & c_l(w) & a_l(w) & b_l(w)
\end{pmatrix}
\]

and \( c_0(w) = 0, \ b_0(w) = 0 \)

(ii) We define scalars

\[ c_i(w), a_i(w), b_i(w) \]

by

\[
A^\# = \begin{pmatrix}
  a_0^*(w) & b_0^*(w) \\
  c_1^*(w) & a_1^*(w) & b_1^*(w) \\
  & \ddots & \ddots \\
  & & c_l^*(w) & a_l^*(w) & b_l^*(w)
\end{pmatrix}
\]

and \( c_0^*(w) = 0, \ b_0^*(w) = 0 \)
LEM 100

(i) \( b_i(w) \rightarrow 0, b_i^*(w) \rightarrow 0 \) \( \text{as} \) \( i \rightarrow \infty \)

(ii) \( c_i(w) \rightarrow 0, c_i^*(w) \rightarrow 0 \) \( \text{as} \) \( i \rightarrow \infty \)

\text{Pf} \quad \text{follows from L89 (i) and L90 (ii)}

By L98 (i), (iii)

(i) \( c_i(w) + a_i(w) + b_i(w) = \theta_e \) \( \text{as} \) \( i \rightarrow \infty \)

(ii) \( c_i^*(w) + a_i^*(w) + b_i^*(w) = \theta_r^* \) \( \text{as} \) \( i \rightarrow \infty \)
LEM 101

(i) \( a_i(w), b_i(w), c_i(w) \in \mathbb{R} \quad \text{signed} \)

(ii) \( a_i^*(w), b_i^*(w), c_i^*(w) \in \mathbb{R} \quad \text{signed} \)

PF (i) \( a_i(w) \in \mathbb{R} \) since it is an eigenvalue.

For real symmetric matrix \( E_r, A E_r^* \)

Now

\[ b_i(w) + c_i(w) = \theta_i - a_i(w) \in \mathbb{R} \]

So \( b_i(w) \in \mathbb{R} \) and \( c_i(w) \in \mathbb{R} \) \( \to b_i(w) \in \mathbb{R} \quad \text{signed} \)

Also \( |b_i(w)| \in \mathbb{R} \)

since \( MN \) is signed real symmetric matrix

\( E_r^*, A E_r^* A E_r^* \)

So

\[ b_i(w) \in \mathbb{R} \quad c_i(w) \in \mathbb{R} \quad 0 \in \mathbb{R} \quad \text{signed} \]

(ii) \( \sim. \)
LEM 101

Let \( \{ w_i \}_{i=0}^d \) denote a st. basis for \( W \). Then

\[ \{ w_i \}_{i=0}^d \text{ are unit orthogonal and} \]

\[ \| w_i \| = \left\| \frac{b_0(w) w_0 + b_1(w) w_1 + \cdots + b_i(w) w_i}{c_i(w) + c(w) w_i} \right\| \quad 0 \leq i \leq d \]

Pf. the \( \{ w_i \}_{i=0}^d \) are unit orthogonal

\[ V = E_0^* V + \cdots + E_d^* V \quad \text{(orthog. ds)} \]

Also for \( 0 \leq i \leq d-1 \)

\[ \langle A w_i, w_i \rangle = \langle w_i, A w_i \rangle \]

\[ \langle c_i(w) w_i + a_i(w) w_i + b_i(w) w_i, w_i \rangle \]

\[ c_i(w) \| w_i \| \]

\[ \langle w_i, c_i(w) w_i + a_i(w) w_i + b_i(w) w_i \rangle \]

\[ b_i(w) \| w_i \| \]

result follows by induction

\[ \Omega \]

\[ \bullet \] Sim result holds for dual st. basis
If $F = \mathbb{R} \times C$ and $\mathbb{D} \mathbb{R} G = (X(\mathbb{R})) \text{ diam } \mathbb{D} \geq 2$

Assume $\{ E : \gamma = 0 \}$ is $G$-poly.

Fix $x \in X$ and let $T = T(x)$ etc.

Until further notice, $W$ is a thin convex $T$-module.

Endpt $r$, dual endpt $b$, and $d$.

Fix $x \in E \ast W$,

Yields standard basis

$\{ E^{\ast} : \gamma = 0 \}$

and dual standard basis

$\{ E^{\ast} : \gamma = 0 \}$

For $W$.

Last time we saw

$$\| E^{\ast} \|^{2} = \| E \|^{2} \frac{b_{0}(w) b_{1}(w) \ldots b_{i}(w)}{c_{1}(w) c_{2}(w) \ldots c_{i}(w)} \quad 0 \leq i \leq d$$

e tc
DEF 103 \hspace{1cm} \forall \alpha \in \mathbb{C} \quad \text{def}

\quad k_i (w) = \frac{b_0 (w) b_1 (w) \ldots b_{i-1} (w)}{c_0 (w) c_1 (w) \ldots c_i (w)}

\quad k_i ^* (w) = \frac{b_0 ^* (w) b_1 ^* (w) \ldots b_{i-1} ^* (w)}{c_0 ^* (w) c_1 ^* (w) \ldots c_i ^* (w)}

---

The following scalar will be useful.

LEM 104 \hspace{1cm} \exists \text{unique scalar } \nu = \nu (w) \text{ such that}

\quad \nu \in W,

\quad \forall E_r ^* E_t \quad E_r ^* = E_r ^*

\quad \forall E_t \quad E_t E_r ^* = E_t

\hspace{1cm} \text{Monom } \forall E \in R

PF \hspace{1cm} \text{We denote the trace of } E_r ^* E_t \text{ on } W

\hspace{1cm} \text{claim}

\quad E_r ^* E_t E_r ^* = \nu E_r ^*

\quad \text{on } W \hspace{1cm} \text{By L89 (iv)}

\quad \eta \in \mathbb{C} \hspace{1cm} \text{By L89 (iv)}

\quad E_t E_r ^* W = E_t W
By L90 (iv)

\[ E_r \times E_t \times W = E_r \times W \]

So

\[ E_r \times E_t \times E_r \times W = E_r \times W \]

Since \( E_r \times W \) has dim 1 \( \exists 0 \neq x \in F \) so

\[ E_r \times E_t \times E_r \times = x \times E_r \]

on \( W \). Take the trace on \( W \) and use \( \text{dim } E_r \times W = 1 \)

to get \( x = m \). Claim proved

We similarly have

\[ E_t \times E_r \times E_t = m \times E_t \]

\( v = m^{-1} \) Obs \( v \in R \) since \( E_r \times E_t \) are real C1
LEM 105 \( \forall \alpha \in i \in d \)

\[
\begin{align*}
(i) \quad \| E_r \gamma \|_2^2 &= k_i(w) \nu \| \gamma \|_2^2 \\
(ii) \quad \| E^\perp \gamma^x \|_2^2 &= k_i^* (w) \nu \| \gamma^x \|_2^2
\end{align*}
\]

pf (i) We saw

\[
\| E_r \gamma \|_2^2 = k_i (w) \| E_\perp \gamma \|_2^2
\]

Also

\[
\| E_\perp \gamma \|_2^2 = \begin{align*}
\langle E_r^\perp, E_r^\perp \gamma \rangle \\
= \langle \gamma, \nu E_\perp E_r^\perp \gamma \rangle \\
= \langle \gamma, \nu \gamma \rangle \\
= \nu \| \gamma \|_2^2
\end{align*}
\]

Ciel Sim.   \( \Box \)
LEM 106  We have

(i) \( E_r \gamma = \frac{\langle \gamma, \gamma^* \rangle \gamma^*}{\| \gamma^* \|^2} \)

(ii) \( E_b \gamma^* = \frac{\langle \gamma^*, \gamma \rangle \gamma}{\| \gamma \|^2} \)

(iii) \( \nabla |\langle \gamma, \gamma^* \rangle|^2 = \| \gamma \|^2 \| \gamma^* \|^2 \)

PF (i) \( \gamma, E_r \gamma \) each nonzero vector in 1-dim space

\( E_r^W = \sum_{j \in E} E_r^j \)

\( E_r \gamma = \alpha \gamma^* \)

\( \alpha > 0 \)

to get \( \alpha \) take inner product of each side with \( \gamma^* \)

(iii) Sim.

(iii) Use (i), (iii) and

\( \nabla E_r^W E_b \gamma^* = E_r^* \) on \( W \)
LEM 107  We have

\[(\cdot) \quad v = \sum_{i=0}^{d} k_i(w)\]

\[(\cdot \cdot) \quad v = \sum_{i=0}^{d} k_i^*(w)\]

\[\text{Proof (\cdot) \quad \eta = \sum_{j=0}^{0} E_j^x \eta = \sum_{i=0}^{d} E_{rii}^x \eta}\]

\[\| \eta \|^2 = \sum_{i=0}^{d} \| E_{rii}^x \eta \|^2 = \| \eta \|^2 \sum_{i=0}^{d} k_i^*(w) v^2\]

by L105 (\cdot).  \hspace{1cm} \text{Result follows.}\]

(\ddot{\cdot})  \text{Sim}
Def 10.8 For \( 0 \leq i \leq d \) we define only \( V_i = v_i(w) \) on \( \text{IF}(x) \) by

\[
v_0 = 1
\]

\[
\lambda v_i = b_{i-1}(w) v_{i-1} + a_i(w) v_i + c_{i-1}(w) v_{i-1}
\]

where \( v_{-1} = 0, \quad c_{i-1}(w) = 1. \)

Observe \( v_i \) has degree \( i \).

(The \( v_i \) are \( \text{sim} \) defined.)
LEM 109

(i) \( V_i(A) \quad E_r^\gamma = E_{r+1}^\gamma \quad (0 \leq i \leq d) \)

(ii) \( V_{dn}(A) \quad W = 0 \)

(iii) \( V_i^*(A^*) \quad E_{b+r}^x = E_{b+r+1}^x \quad (0 \leq i \leq d) \)

(iv) \( V_{dn}^*(A^*) \quad W = 0 \)

Pf (i), (ii) Abbr \( A_i = E_{r+1}^\gamma \) in \( 0 \leq i \leq d \).

\[ B_r \text{ omit} \]

\[ AA_i = b_i(w)a_i + q_i(w)a_i \quad (0 \leq i \leq d) \]

Comparing this with Def 108 gives

\[ V_i(A) a_0 = a_i \quad (0 \leq i \leq d) \]

But \( \sum a_i = 0 \) as

\[ V_{dn}(A) a_0 = 0 \]

Now \( V_{dn}(A) W = V_{dn}(A) M a_0 \)

\[ = M \frac{V_{dn}(A) a_0}{0} \]

\[ = 0 \]

(iii), (iv) Sim
LEM 110

(i) \( V_{1n} \) is a non-zero scalar multiple of the null space of \( A \) on \( W \).

(ii) The norm of \( V_{1n} \) are \( \{ \theta_n \}_{i=0}^d \).

(iii) \( V_{1n} \) is a non-zero scalar multiple of the null space of \( A^* \) on \( W \).

(iv) The norm of \( V_{1n} \) are \( \{ \theta_n \}_{i=0}^d \).

PF (i) Clear from L109 (i), (iii)

(iii) The eigenspace of \( A \) on \( W \) are:

\[ \mathbb{E}_{\theta_i} W = \{ \theta_i \} \]

(iii), (iv) Similar.
the \( v_i(W), v_i^*(W) \) are normalized as follows.

**LEM III  F_{n \in \mathbb{N}}**

(i) \( v_i(\emptyset) = k_i(W) \)

(ii) \( v_i^*(\emptyset) = k_i^*(W) \)

**Proof (i)**

\[
\begin{align*}
 k_i(W) \| E_{r^i}^* \|^2 &= \| E_{r^i}^* \|^2 \\
 &= \left< \gamma, E_{r^i}^* \right> \\
 &= \left< \gamma, v_i(A) E_{r^i}^* \right> \\
 &= \left< v_i(A) \gamma, E_{r^i}^* \right> \\
 &= v_i(\emptyset) \left< \gamma, E_{r^i}^* \right> \\
 &= v_i(\emptyset) \| E_{r^i}^* \|^2 \\
\end{align*}
\]
We now give the transition matrix between the standard and dual standard bases of $W$.

**Thm 112**

$F_n$ osc and

\[(\star) \quad E_{r+c}^* \gamma = \frac{\langle \gamma, \gamma^* \rangle}{\| \gamma^* \|^2} \sum_{c=0}^{d} v_{c} (\theta_{c} + c) \quad E_{r+c}^* \gamma^* \]

\[(\star\star) \quad E_{r+c} \gamma = \frac{\langle \gamma^* \gamma \rangle}{\| \gamma \|^2} \sum_{c=0}^{d} v_{c}^* (\theta_{c} + c) \quad E_{r+c}^* \gamma^* \]

**Pf (\star)\**

$E_{r+c}^* \gamma = v_{c} (A) \quad E_{r+c}^* \gamma^*$

$= (E_0 + \cdots + E_0) v_{c} (A) \quad E_{r+c}^* \gamma^* \]

$= \sum_{c=0}^{d} E_{r+c} \quad v_{c} (A) \quad E_{r+c}^* \gamma^* \]

$= \sum_{c=0}^{d} v_{c} (\theta_{c} + c) \quad E_{r+c} \quad E_{r+c}^* \gamma^* \frac{\langle \gamma, \gamma^* \rangle}{\| \gamma^* \|^2} \quad \text{by LOC}$

\[(\star\star) \quad \text{Sim.} \]
DEF 12. For $0 \leq i \leq d$ we define polynomials $u_i = u_i(w)$ in $\mathbb{F}[w]$, so

$$u_0 = 1$$

$$\lambda u_i = c_i(w) u_{i-1} + a_i(w) u_{i+1} + b_i(w) u_i \quad (0 \leq i \leq d-1)$$

where $u_{-1} = 0$.

We observe $u_i$ has degree $i$ for $0 \leq i \leq d$.

(* poly $u_i$ s in defined *)

LEM 12. We have $\lambda u_i \equiv 0 \pmod{d}$

(\ref{1}) \quad $u_i = \frac{v_i}{k_i(w)}$

(\ref{2}) \quad $u_i^* = \frac{v_i^*}{k_i^*(w)}$

Pf. ex. \hfill $\square$
the $u_1, u_2$ are normalized as follows.

**LEMMA 118** Fa ose de

1. $u_c(r_e) = 1$
2. $u_c(r_0) = 1$

**Proof** By L113, L111. □
Theorem 11 \( \text{Lin} \)

\[ \langle \mathcal{E}_{\tau}, \mathcal{E}_{\tau}^* \rangle = k^*_{\tau}(\omega) k_{\tau}^*(\omega) u^*_{\tau}(\omega_{t+1}) \langle \tau, \tau^* \rangle \]

\[ \langle \mathcal{E}_{\tau}, \mathcal{E}_{\tau}^* \rangle = k^*_{\tau}(\omega) k_{\tau}^*(\omega) u^*_{\tau}(\omega_{t+1}) \langle \tau, \tau^* \rangle \]

\[ u_{\tau}(\omega_{t+1}) = u^*_{\tau}(\omega_{t+1}) \quad (\text{Askey-Wilson duality}) \]

Proof (i)

\[ \langle \mathcal{E}_{\tau}, \mathcal{E}_{\tau}^* \rangle = \left( \frac{\langle \tau, \tau^* \rangle}{\| \tau^* \|_2} \right) \sum_{h=0}^{d} \left( \frac{\langle v_{\tau}(\omega_{t+h}) \mathcal{E}_{t+h} \tau^*, \mathcal{E}_{t+h} \tau^* \rangle}{\| \mathcal{E}_{t+h} \tau^* \|_2^2} \right) \]

\[ = \left( \frac{\langle \tau, \tau^* \rangle}{\| \tau^* \|_2^2} \right) \frac{\| \mathcal{E}_{t+h} \tau^* \|_2^2}{k_{\tau}(\omega) u^*_{\tau}(\omega_{t+h}) k_{\tau}^*(\omega) v^{-1} \| \tau^* \|_2^2} \]

(ii) \( \sum \)

(iii) \( B_i \) (i, ii) \( \Box \)
Here is the or-magnitude $f$ the $v_i$.

Th 117 \[ F_\alpha \quad 0 = i, \beta \leq D \]

(\text{i}) \[ \sum_{h=0}^{d} v_i(\theta + h) v_j(\theta + h) k_h^*(w) = \delta_{ij} v k^*_\alpha(w) \]

(\text{ii}) \[ \sum_{h=0}^{d} v_h(\theta + h) v_h(\theta + h) k_h^*(w) = \delta_{ij} v(k^*_\alpha)(w) \]

Pf Use Thm 112.
Here is the setup for the \( \psi_i \):

**Theorem 118** For oscillated

(i) \[
\sum_{h=0}^{d} u_i(\theta_{eh}) u_j(\theta_{eh}) k_{i}^*(w) = \delta_{ij} \vee k_i(w) \]

(ii) \[
\sum_{h=0}^{d} u_i(\theta_{eh}) u_i(\theta_{eh}) k_{h}(w) = \delta_{ii} \vee k_i^*(w) \]

**Proof** Use Thm 117 and def of \( \psi_i \). \( \square \)
Here is the orthogonality for the $u^*_i$

**Theorem 119** For $0 \leq i_1, i_2 \leq d$

(i) \[ \sum_{h=0}^{d} u_{i_1}^* (\theta_{c+h}^*) u_{i_2}^* (\theta_{c+h}^*) k_h(w) = \delta_{i_1}(w) k_i^*(w)^{-1} \]

(ii) \[ \sum_{h=0}^{d} u_{i_1}^* (\theta_{c+r}^*) u_{i_2}^* (\theta_{c+r}^*) k_h^*(w) = \delta_{i_2}(w) k_i^*(w)^{-1} \]

**Proof** Use Th 118 and AW duality \( \square \)
Here is the orthogonality for the $v_i^*$

**Lemma 120** For $0 \leq i, j \leq d$

(i) \[ \sum_{h=0}^{d} v_i^*(\theta_{rh}) v_j^*(\theta_{rh}) k_h(w) = \delta_{ij} k_i^*(w) \]

(ii) \[ \sum_{h=0}^{d} v_i^*(\theta_{rh}) v_h^*(\theta_{rh}) k_h^*(w) = \delta_{ij} k_i^*(w) \]

**Proof** Use Thm 119 and $v_i^* = k_i^*(w) u_i^*$
Given $PRG$, \( T = (X \times R) \) diam $D \geq 2$

Assume \( \{ E : 3 \} \) is poly

Fix $x \in X$ write $T = T(x)$ etc

Fix a thin row $T$ module $W$ entry $i$, dual entry $t$

diam $d$. Fix $o \neq i \in E', W$, $o \neq t \in E_t W$

Find $a_i(w)$, $b_i(w)$, $c_i(w)$ etc.

by L98:

\[
\begin{align*}
  c_i(w) + a_i(w) + b_i(w) &= \theta_t \\
  c_i^*(w) + a_i^*(w) + b_i^*(w) &= \theta^*_t
\end{align*}
\]

For $d = 0$

\[
\begin{align*}
  a_0(w) &= \theta_t \\
  a^*_0(w) &= \theta^*_t
\end{align*}
\]

nothing more to say.

Until further notice $d \geq 1$. 


We introduce a parametr \( \eta_i = \eta_i(\omega) \)

**LEM 121** \( \exists \ non_0 \ \eta_i = \eta_i(\omega) \in IF \ s.t. \)

\[ (i) \quad b_0(\omega) = \frac{\eta_i}{\theta_{\eta_i} - \theta} \]

\[ (ii) \quad b_0(\omega) = \frac{\eta_i}{\theta_{\eta_i} - \theta} \]

**Pf** the poly \( u_i = u_i(\omega) \) sat

\[ \lambda u_i = c_i(\omega) u_{i-1} + a_i(\omega) u_i + b_i(\omega) u_{i+1} \quad 0 \leq i \leq d \]

So for \( i = 0 \)

\[ \lambda u_0 = a_0(\omega) u_0 + b_0(\omega) u_1 \]

\[ u_1 = \frac{\lambda - a_0(\omega)}{b_0(\omega)} \]

Also

\[ a_0(\omega) + b_0(\omega) = \theta \]

So

\[ u_i = 1 + \frac{\lambda - \theta}{b_0(\omega)} \]
\begin{align*}
v_i &= 1 + \frac{\lambda - \sigma_r^*}{b^*_o(w)} \\
\text{By AW duality,} \\
v_i(\theta_{en}) &= v_i(\theta_r^*) \\
\text{given} \\
\frac{\theta_{en} - \theta_r}{b_o(w)} &= \frac{\theta_{en}^* - \theta_r^*}{b_o^*(w)} \\
\text{so} \\
b_o(w) (\theta_{en} - \theta_r) &= b_o^*(w) (\theta_{en}^* - \theta_r^*) \\
\text{Call this common value } \psi_i \text{ and set } \psi_i = 0 \\
\text{Result follows.} \quad \square
\end{align*}
Lemma 127. We have

(i) \[ a_0(w) = \theta_\ell + \frac{\psi_r}{\sigma_r^* - \psi_i^*}. \]

(ii) \[ a_0^*(w) = \sigma_r^* + \frac{\psi_i}{\sigma_r^* - \psi_i}. \]

(iii) \[ u_r(w) = 1 + \frac{(\lambda - \theta_\ell)(\sigma_m^* - \sigma_r^*)}{\psi_i}. \]

(iv) \[ u_r^*(w) = 1 + \frac{(\sigma_m - \theta_\ell)(\lambda - \sigma_r^*)}{\psi_i}. \]

Proof (i) Use \[ a_0(w) = \theta_\ell - b_0(w). \]

(ii) 

(iii) Use:

\[ u_r = 1 + \frac{\lambda - \theta_\ell}{b_0(w)}. \]

(iv) \[ \Box \]
LEM 123 \[ F_a \]

\[ (i) \quad \varsigma_i (w) (\theta_{ri}^* - \theta_{ri}^{*\prime}) - b_i (w) (\theta_{ri}^{*\prime} - \theta_{rin}^{*\prime}) = \\
(\theta_{rin} - \theta_e) (\theta_{ri}^{*\prime} - \theta_e) + \varphi, \]

\[ (ii) \quad \tilde{\varsigma}_i (w) (\theta_{ri}^* - \theta_{ri}^{*\prime}) - \tilde{b}_i (w) (\theta_{ri}^{*\prime} - \theta_{rin}^{*\prime}) = \\
(\theta_{rin}^{*\prime} - \theta_e) (\theta_{ri}^{*\prime} - \theta_e) + \varphi, \]

pf (i) the role \( u_i = u_i (w) \) sat

\[ \lambda u_i = \varsigma_i (w) u_i + a_i (w) u_i + b_i (w) u_i \]

take \( \lambda = \theta_{rin} \) we AW duality

\[ u_{y_i} (\theta_{rin}) = u_{y_i}^{*\prime} (\theta_{ri}^{*\prime}) = 1 + \frac{(\theta_{rin} - \theta_e) (\lambda - \varsigma_i^{*\prime})}{\varphi}, \]

Also \( \lambda \) lem \( a_i (w) u_i \) use

\[ \varsigma_i (w) + a_i (w) + b_i (w) = \theta_e \]

(iii) sim
Example so far:

For Bip or dual Bip cases,

Assume \( \Gamma \) is Bip

So \( a_i = 0 \quad \text{for} \quad 0 \leq i < D \)

\( E_i^* A E_i = 0 \)

Def

\[
R = \sum_{i=0}^{D-1} E_i^* A E_i \quad \text{"raising matrix"}
\]

\[
L = \sum_{i=0}^{D-1} E_i A E_i \quad \text{"lowering matrix"}
\]

Obs

\( A = R + L \)

\( R E_i^* V \leq E_i^* V \quad (0 \leq i < D) \)

Also

\( E_i^* R = R E_i \quad 0 \leq i < D \)

\( E_i^* L = L E_i \quad 0 \leq i < D \)
For \( R \) bip, we show each \( h \) is an \( \mathfrak{u} \) submodule.

**LEM 124** For bip \( k \), let \( W \) denote an \( \mathfrak{u} \) module with dual vector \( W \). Fix any \( \psi \in E_0 W \). Then

\[(i) \quad R \mathfrak{E}^* \psi + L \mathfrak{E} \psi = \theta \mathfrak{E} \psi \quad 0 \leq i \leq 0 \]

\[(ii) \quad \text{Suppose} \ \gamma \ \text{is an eigenvalue for} \ E_0 A^* E_0. \ \text{Then} \]

\[\theta \mathfrak{E} \psi R \mathfrak{E} \psi + \theta \mathfrak{E} L \mathfrak{E} \psi = \]

\[\left( \theta \mathfrak{E}_0 \theta \mathfrak{E}_0 - \lambda \theta \mathfrak{E}_0 + \kappa \theta \mathfrak{E}_0 \right) \mathfrak{E} \psi \quad 0 \leq i \leq 0 \]

where \( \lambda \) is the eigenval of \( E_0 A^* E_0 \) and

\[\theta \mathfrak{E}_0, \ \theta \mathfrak{E}_0, \ \theta \mathfrak{E}_0 \ \text{are inlets}.

**PF (i)**

\[R \mathfrak{E} \psi + L \mathfrak{E} \psi = \mathfrak{E} \psi \mathfrak{R} \psi + \mathfrak{E} \psi \mathfrak{L} \psi\]

\[= \mathfrak{E} \psi (\mathfrak{R} + \mathfrak{L}) \psi\]

\[= \mathfrak{E} \psi A \psi\]

\[= \mathfrak{E} \psi \mathfrak{E}^* \psi\]
Recall

\[ A^* E_\psi W = E_\psi W + B E_\psi W \]

\( \gamma \) is equal to \( E_\psi A^* E_\psi \) with a small \( \delta \) so

\[ (A^* - \alpha I) \gamma \in E_\psi W \]

So

\[ (A - \delta E_\psi I)(A^* - \alpha I) \gamma = 0 \]

So

\[ \theta_{tm} R E_{\psi m} \gamma + \theta_{tm} L E_{\psi m} \gamma \]

\[ = (R E_{\psi m} + L E_{\psi m}) A^* \gamma \]

\[ = (E_{\psi m} R + E_{\psi m} L) A^* \gamma \]

\[ = E_{\psi m} A A^* \gamma \]

\[ = E_{\psi m} A (A^* - \alpha I) \gamma + \alpha E_{\psi m} A \gamma \]

\[ = \theta_{tm} E_{\psi m} (A^* - \alpha I) \gamma + \theta_{tm} \alpha E_{\psi m} \gamma \]

\[ = (\theta_{tm} E_{\psi m} - \theta_{tm} \alpha + \theta_{tm} \alpha) E_{\psi m} \gamma \]
LEM 125  For bipartite each $\mathcal{M}$

$T$-module is thin.

Pf. let $W$ denote an $\mathcal{M}$-valued $T$-module.

With content $\varepsilon$. By const $E\varepsilon W$ is nonzero

and invariant under $E\varepsilon A^r E\varepsilon$

$E\varepsilon A^r E\varepsilon$ is real symmetric so it is diagonalizable

on $E\varepsilon W$. So if $0 \neq \gamma \in E\varepsilon W$ that is an

eigenvalue for $E\varepsilon A^r E\varepsilon$. By L124

$$ RE^{i\gamma} + LE^{i\gamma} \in \text{Span} \left( E^{i\gamma} \right) $$

$$ SE^{i\gamma} + LE^{i\gamma} \in \text{Span} \left( E^{i\gamma} \right) $$

for $0 \leq i \leq 0$, where $0^+$ for index

Since $\left\{ E^{i\gamma} \right\}$ are mutually orthogonal,

$$ RE^{i\gamma} \in \text{Span} \left( E^{i\gamma} \right) \quad (0 \leq i \leq b) $$

$$ LE^{i\gamma} \in \text{Span} \left( E^{i\gamma} \right) \quad (0 \leq i \leq d) $$
def $W' = \text{Span}\{ E^x_i \mid 0 \leq i \leq 3 \}$

So $W' \leq W$

clear $W' = W$

Obs $W' \neq 0$ since

$0 \neq 0 = \sum_{i=0}^{3} E^x_i \in W'$

By construct $A^x W' \leq W'$

By construct $R W' \leq W'$, $L W' \leq W'$

and $A = R + L$ so

$A W' \leq W'$

Now $W'$ is $T$-module.

Now $W' = W$ by the kernel of $W$.

Now $\mu_0 = 0$

$E^x W = \text{Span}\{ E^x_i \}$

has dim $\leq 1$ so $W$ is thin. \square
LEM 126  Assume the  copoly structure $E_i S_i^a$ is dual bip. then each unit T-module is thin.

PF. Very similar to pt fn bip just replace

$A \cap i^a, E_i \& E_i^b$ everywhere.  \text{ } \Box