0. Introduction.

The well-known Birch and Swinnerton-Dyer conjecture gives a deep connection between the leading coefficient of the \( L \)-series and the arithmetic properties of an abelian variety. Both are very important and subtle. This paper is part of an effort to compute the analytic side explicitly in a special case. Indeed, we are interested in the central derivative of certain algebraic Hecke \( L \)-series, related to CM abelian varieties or more precisely pieces of it (CM motives). The result, together with the Gross-Zagier formula proved by Zhang ([Zh]), would also give a new way to compute the height of certain Heegner cycles on a Kuga-Sato variety.

Let \( p \equiv 3 \mod 4 \) be a prime number such that \( p > 3 \). Let \( k \geq 0 \) be an integer. Let \( E = \mathbb{Q}(\sqrt{-p}) \) and view it as a subfield of \( \mathbb{C} \) such that \( \sqrt{-p} = i \sqrt{p} \). Let \( h_p \) be the ideal class number of \( E \). A canonical Hecke character of \( E \) of weight \( 2k + 1 \) is a Hecke character \( \mu \) satisfying

1. The conductor of \( \mu \) is \( \sqrt{-p} \mathcal{O}_E \).
2. \( \mu(\mathfrak{A}) = \overline{\mu(\mathfrak{A})} \) for an ideal \( \mathfrak{A} \) relatively prime to \( \sqrt{-p} \mathcal{O}_E \).
3. \( \mu(\alpha \mathcal{O}_E) = \pm \alpha^{2k+1} \).

There are \( h_p \) such Hecke characters, differing from each other by an ideal class character. A canonical Hecke character of weight 1 is simply called a canonical character of \( E \).

Canonical Hecke characters of \( E \) of weight one are very closely related to the elliptic curves \( A(p) \) studied by Gross in [Gro], while Canonical Hecke characters of weight \( 2k + 1 \) are closely related to a piece of the \((2k+1)\)-th symmetric power of \( A(p) \). The second condition implies that the root number of \( \mu \) is \( \pm 1 \), given by \( \left( \frac{2}{p} \right) (-1)^k \) in the canonical case. When the root number is one, the leading
coefficient of the L-series is usually the special value. In the canonical case, the special value was proved nonvanishing ([MR]) and given by a beautiful formula discovered by Rodriguez Villegas ([RV]). The explicit formula was generalized by the author using a different method ([Ya1]). When the root number is $-1$, the special value at the center is automatically zero, and the leading coefficient is most likely the central derivative. In fact, when $k = 0$, S. Miller and the author ([MY]) have just proved that for all $p \equiv 3 \text{mod } 8$, the central derivative $L'(1, \mu) \neq 0$. In this paper, we will give an explicit formula to compute the central derivative when $p \equiv 3 \text{mod } 8$ and $k$ is even. The other case where $p \equiv 7 \text{mod } 8$ and $k$ is odd can be done similarly. In fact, the method works for its quadratic twists too, but we stick to one special case to keep it simple.

From now on, we assume that $p \equiv 3 \text{mod } 8$ and $k \geq 0$ is an even integer. So the root number of $\mu$ is $(−1)^k(\frac{2}{p}) = −1$, and the central L-value $L(k + 1, \mu) = 0$ automatically. The purpose of this paper is to give a formula for the central derivative $L'(k + 1, \mu)$. In fact, our formula is about the partial L-series. For each ideal class $C$ of $E$, we can define the partial L-series

$$L(s, \mu, C) = \sum_{\mathfrak{a} \in C, \text{ integral}} \mu(\mathfrak{a})(N\mathfrak{a})^{-s}. \tag{0.1}$$

Obviously

$$L(s, \mu) = \sum_{C} L(s, \mu, C), \quad \text{and} \quad L(s, \mu \xi, C) = \xi(C)L(s, \mu, C), \tag{0.2}$$

where $\xi$ is an ideal class character of $E$. Moreover, $L(s, \mu, C)$ is characterized by (0.2).

For an ideal class $C$, we choose a primitive integral ideal $\mathfrak{a}$ in $C$ relatively prime to $2p$. Write

$$\mathfrak{a} = [a, b + \sqrt{-p}] \tag{0.3}$$

with $a = N\mathfrak{a} > 0$ and $b^2 \equiv -p \text{mod } 4ap$. Let $\tau_{\mathfrak{a}} = \frac{b + \sqrt{-p}}{2a}$.

Let $\epsilon = (−p, \mathfrak{a})_h$ be the quadratic Hecke character of $\mathbb{Q}$ associated to $E/\mathbb{Q}$ by class field theory. Let

$$\Lambda(s, \epsilon) = L(s, \epsilon_\infty)L(s, \epsilon), \quad \text{where} \quad L(s, \epsilon_\infty) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right). \tag{0.4}$$

Define $\rho(n)$ for an integer $n > 0$ via

$$\zeta_E(s) = \sum_{n=1}^{\infty} \rho(n)n^{-s}. \tag{0.5}$$
where \( \zeta_E \) is the Dedekind zeta function of \( E \). Finally we define two functions

\[
C_k(t) = \sum_{m=0}^{k} \binom{k}{m} \frac{t^m}{m!},
\]

and for \( t > 0 \)

\[
\beta_k(t) = \int_{1}^{\infty} e^{-tu} (u-1)^k u^{-k-1} du.
\]

They are two ‘basic’ solutions of the differential equations

\[
tC''(t) + (1 + t)C'(t) - kC(t) = 0.
\]

**Theorem 0.1.** Let the notation be as above. Then

\[
L'(k+1, \mu, C) = \frac{\pi \mu(\mathfrak{A})}{\sqrt{\beta} \alpha^{k+1}} \left[ h_p(\log \frac{p}{2a} + 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)} + \sum_{j=1}^{k} \frac{1}{j})
\right.
\]

\[
-2 \sum_{n>0} a_n C_k(-\frac{2\pi n}{a \sqrt{p}}) e^{\frac{2\pi i n \tau A}{p}}
\]

\[
-2 \sum_{n<0} \rho(-n) \beta_k(-\frac{2\pi n}{a \sqrt{p}}) e^{\frac{2\pi i n \tau A}{p}}
\]

Here

\[
a_n = \sum_{q \text{ inert}} (\text{ord}_q n + 1) \rho(n/q) \log q + (\text{ord}_p n + \frac{1}{2}) \rho(n) \log p.
\]

The sum \( \sum_{j=1}^{k} \frac{1}{j} \) should be treated as zero when \( k = 0 \).

We remark that the sum in the bracket is the special value of a nonholomorphic modular form at a CM point \( \tau_A \). We also remark that at most one term in \( a_n \) is nonzero and that \( a_n + \frac{1}{2} \rho(n) \log p \) has an interesting arithmetic interpretation ([KRY, section 5]). When \( C \) is trivial, we can do a little bit more.

**Theorem 0.2.** Let \( \mu \) be a canonical Hecke character of weight one. Then the partial central derivative is

\[
L'(1, \mu, \text{trivial})
\]

\[
= \frac{\pi}{\sqrt{p}} \left[ h_p(\log \frac{p}{2} + 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)})
\right.
\]

\[
+ 2 \sum_{n>0} (-1)^{n-1} a_n e^{-\frac{\pi n}{\sqrt{p}}}
\]

\[
2 \sum_{n<0} (-1)^{n-1} \rho(-n) \beta_0(-\frac{2\pi n}{\sqrt{p}}) e^{-\frac{\pi n}{\sqrt{p}}}
\]

\[
\left].
\right.
\]
and

\[ L'(1, \mu, \text{trivial}) = \frac{4\pi}{\sqrt{p}} \left[ h_p(\log p + 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)}) \right. \]

\[ \left. - 2 \sum_{n>0} a_n e^{-\frac{4\pi n}{\sqrt{p}}} \right. \]

\[ \left. - 2 \sum_{n<0} \rho(-n) \beta_0(-\frac{8\pi n}{\sqrt{p}}) e^{-\frac{4\pi n}{\sqrt{p}}} \right. \]

\[ \left. -2 \log 2 \sum_{n>0} \rho(n) e^{-\frac{2\pi n}{\sqrt{p}}} \right] \]

This paper is organized as follows. In section 1, we use a result in [Ya1] to write the Hecke L-series at the sum of certain ‘incoherent’ Eisenstein series valued at some CM points. In section 2, we prove some preliminary local results for section 3. In section 3, we used Kudla’s idea ([Ku]) to compute the central derivative of the Eisenstein series in section 1 and prove theorem 0.1. It might be worthwhile to point out that sections used in the Eisenstein series are not standard at bad primes. The same phenomenon also occurs in Kudla and Rapoport’s work in a higher dimension case ([KR]). In section 4, we prove theorem 0.2. In the process, we also prove a functional equation of the Eisenstein series involved with respect to \( \tau \) (instead of \( s \)).

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1. **Set-up.**

Let \( W = E \) with the skew-Hermitian form \( <x, y> = \delta xy \), and let \( W + W_- \) be its doubling. Let \( G = G(W) = U(1) = E^1 \) and \( H = G(W + W_-) = U(1, 1) \) be the corresponding isometry groups acting on the right. Then one has a canonical embedding

\[(1.1) \quad i : G \times G \rightarrow H, \quad (x_1, x_2)i(g_1, g_2) = (x_1g_1, x_2g_2). \]

Let \( W^d = \{(w, w) : w \in W\} \) and \( W_d = \{(w, -w) : w \in W\} \). Then \( W + W_- \) has the standard complete polarization

\[ W + W_- = W_d \oplus W^d, \]
and the standard $E$-basis $e = \frac{1}{\sqrt{d}}(1, -1)$ and $f = (1, 1)$. With respect to the standard basis, the map (1.1) is given by

$$i(g_1, g_2) = \frac{1}{2} \left( \begin{array}{cc} g_1 + g_2 & \frac{1}{\sqrt{d}}(g_1 - g_2) \\ 2\delta(g_1 - g_2) & g_1 + g_2 \end{array} \right).$$

We will write $i(g)$ for $i(g, 1)$ in this paper.

Let $P$ be the stabilizer of $W^d$ in $H$ (the standard Siegel parabolic subgroup of $H$). Then $P$ has the Levi decomposition $P = NM$ where, with respect to the standard complete polarization,

$$N = \{ n(b) = \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) : b \in \mathbb{Q} \},$$

$$M = \{ m(a) = \left( \begin{array}{cc} a & 0 \\ 0 & \bar{a}^{-1} \end{array} \right) : a \in E^{*} \}.$$

One has

$$H = P \times i(G).$$

Let $\chi_{\text{can}}$ be a fixed canonical Hecke character of $E$ (of weight 1), and let $\chi = \chi_{\text{can}}|_{\mathbb{A}}^{1/2}$ be its unitary counterpart. Then there is a unique character $\eta$ of $G_{\mathbb{A}}$ such that $\mu = \chi_{\bar{\eta}}|_{\mathbb{A}}^{-k-\frac{1}{2}}$, where $\bar{\eta}(z) = \eta(z/\bar{z})$ is a Hecke character of $E$. So

$$L(s + k + 1, \mu, C) = L(s + \frac{1}{2}, \chi_{\bar{\eta}}, C).$$

Let $I(s, \chi) = \text{Ind}_{P(H_{\mathbb{A}})}^{H(\mathbb{A})} \chi|_{\mathbb{A}}^{1/2}$ be the degenerate (induced from a character of a maximal parabolic subgroup) principal series representation. Given a function $\Phi = \prod \Phi_v \in I(s, \chi)$, one defines the Eisenstein series

$$E(h, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} \Phi(\gamma h, s)$$

where $h \in H(\mathbb{A})$. It is absolutely convergent for $\text{Re } s >> 0$ and has a meromorphic continuation to the whole complex $s$-plane. We normalize the Eisenstein series

$$E^{*}(h, s) = \Lambda(2s + 1, \epsilon)E(h, s).$$

Let $\Phi = \prod \Phi_q \in I(s, \chi)$ be the section $\Phi_{\bar{q}}$ defined in [Ya1, Theorem 1.11]. More specifically, when $q$ is nonsplit, $q$ is defined in [Ya1, Theorem 1.11].

$$\Phi_q(n(b)m(a)i(g)) = \chi_q(a)|a\bar{a}|^{s+1}\bar{\eta}(g).$$

When $q$ is split, $\Phi_q$ is characterized by (after identifying $G_q$ with $\mathbb{Q}_q^{*}$ as in [Ya1, section 1.2])

$$\Phi_q(i(g)) = \chi_q(g)|g|^{s+\frac{1}{2}} \text{ char}(\mathbb{Z}_q)(g) + \chi_q(g)|g|^{-s-\frac{1}{2}} \text{ char}(q\mathbb{Z}_q)(1/g).$$

Since we will stick to the same $\Phi$ throughout this paper, we will drop $\Phi$ in the notation from now on.
Proposition 1.1. For each ideal class $C$, choose an idele $a_C \in E_\mathfrak{A}^*$ such that the corresponding ideal is $\mathfrak{A} \in C$, primitive and relatively prime to $2p$. Let $g_C$ be the image of $a_C$ in $G_\mathfrak{A}$ under the map $z \mapsto z/\overline{z}$. Then

$$L'(k + 1, \mu) = \frac{\pi}{2} \sum_{C \in \text{CL}(E)} \tilde{\eta}(\mathfrak{A}) E^*(i(g_C), 0).$$

Proof. Locally, we choose the Haar measure on $G_q$ such that $\text{meas}(G_q) = 1$ when $q$ is nonsplit and $\text{meas}(\mathbb{Z}_q^*) = 1$ when $q$ is split. We take product measure on $G(\mathbb{A})$ and quotient measure on $[G] = G(\mathbb{Q}) \backslash G(\mathbb{A})$. Then [Ya1, Theorem 1.1] implies

$$L(s + k + 1, \mu) L(2s + 1, \epsilon) = \frac{1}{\text{meas}[G]} \int_{[G \times G]} E(i(g_1, g_2), s) \eta(g_1)(\chi \eta)^{-1}(g_2) dg_1 dg_2.$$

Since $E(i(g_1, g_2), s) = E(i(g_1 g_2^{-1}, 1), s) \chi(g_2)$, a substitution gives

$$L(s + k + 1, \mu) L(2s + 1, \epsilon) = \int_{[G]} E(i(g), s) \eta(g) dg.$$

Next, notice that $\prod g_C^{-1} U^1$ is a fundamental domain for $[G]$ (see [RVY, proof of Proposition 1.3]) where

$$U^1 = \prod_{q \text{ inert}} E_q^1 \times \prod_{q \text{ split}} \mathbb{Z}_q^* \times \{g \in E_p^1 : g \equiv 1 \mod \delta\}.$$ 

So

$$L(s + k + 1, \mu) L(2s + 1, \epsilon) = \sum_C \int_{g_C^{-1} U^1} E(i(g), s) \eta(g) dg = \sum_C \eta(g_C) \int_{U^1} E(i(g_C g), s) \eta(g) dg.$$

Finally, one has by definition,

$$\Phi(hi(g), s) = \tilde{\eta}(g) \Phi(h, s)$$

for all $h \in H(\mathbb{A})$ and $g \in U^1$. Therefore

$$\frac{L(s + k + 1, \mu)}{L(2s + 1, \epsilon)} = \sum_C \tilde{\eta}(\mathfrak{A}) E(i(g_C), s) \int_{U^1} dg = \frac{1}{2} \sum_C \tilde{\eta}(\mathfrak{A}) E(i(g_C), s).$$
Here $\frac{1}{2}$ appears because the subgroup $\{g \in E^1_p : g \equiv 1 \mod \delta\}$ is of index 2 in $G_p = E^1_p$. Multiplying both sides by $\Lambda(2s + 1, \epsilon)$, one has

\[(1.10) \quad L(2s + 1, \epsilon_\infty)L(s + k + 1, \mu) = \frac{1}{2} \sum_C \tilde{\eta}(2\lambda)E^*(i(g_C), s).\]

Recall that $L(1, \epsilon_\infty) = \pi^{-1}$. Taking derivative at $s = 0$, one gets the desired formula.

To motivate the computation in the next section, we recall that

\[(1.11) \quad E^*(h, s) = \sum_{t \in \mathbb{Q}} E^*_t(h, s)\]

where

\[(1.12) \quad E^*_0(h, s) = \Phi^*(h, s) + \prod_q W^*_0, q(h, s)\]

is the constant term, and

\[(1.13) \quad E^*_t(h, s) = \prod_q W^*_t, q(h, s)\]

is the $t$-th (normalized) Fourier coefficient of $E$ for $t \neq 0$. Here $\Phi^*(h, s) = \Lambda(2s + 1, \epsilon)\Phi(h, s)$ and

\[(1.14) \quad W^*_t, q(h, s) = L(2s + 1, \epsilon_q)W_{t, q}(h, s)\]

is the normalized local Whittaker functions (including $t = 0$).

2. Local Whittaker functions.

The purpose of this section is to compute the local Whittaker functions and their central derivative. We will write $W_{t, q}(s)$ for $W_{t, q}(1, s)$ from now on and drop subscript $q$ when no confusion occurs. For example, $\chi$ will mean $\chi_q$ in this section. We start with the nonsplit case.

Lemma 2.1. Assume $q$ is nonsplit in $E$. Then one has for $t \in \mathbb{Q}_q$,

\[W_{t, q}(s) = \int_{\mathbb{Q}_q} (\chi \tilde{\eta})^{-1}(b + \frac{1}{2\delta})|b + \frac{1}{2\delta}|^{- \frac{s-\frac{1}{2}}{2}} \psi(-tb) db.\]

Proof. Since $q$ is nonsplit, $H_q = P_q \times i(G_q)$. So one can write

\[wn(b) = n(x)m(y)i(g)\]
Then
\[ g = \bar{y}/y, \quad \text{and} \quad b + \frac{1}{2\delta} = \bar{y}^{-1}g. \]

So
\[ y = \frac{1}{b + \frac{1}{2\delta}}, \quad g = \frac{b + \frac{1}{2\delta}}{b - \frac{1}{2\delta}}. \]

Now it is clear that
\[
W_{t,q}(s) = \int_{Q_q} \Phi(wn(b))\psi(-tb)db \\
= \int_{Q_q} \chi(y)\bar{y}(\bar{y}/y)|b + \frac{1}{2\delta}|^{-s}e^{-\frac{1}{2}\psi(-tb)}db \\
= \int_{Q_q} (\chi\bar{y})^{-1}(b + \frac{1}{2\delta})|b + \frac{1}{2\delta}|^{-s}e^{-\frac{1}{2}\psi(-tb)}db
\]
as stated.

**Proposition 2.2.** When \( q \nmid 2p\infty \) is inert, \( W^*_{t,q}(s) = 0 \) unless \( t \in \mathbb{Z}_q \). Assume \( t \in \mathbb{Z}_q \), and set \( X = q^{-2s} \). Then

1. \[ W^*_{0,q}(s) = L(2s, \epsilon_q). \]

**Proposition 2.3.**

1. \[ W^*_{0,\infty}(s) = -i2^{2s}p^sL(2s, \epsilon_\infty) \prod_{j=1}^{k} \frac{j - s}{j + s}. \]

2. When \( t > 0 \), one has
\[ W^*_{t,\infty}(0) = -2ie^{-\pi t/\sqrt{p}}C_k\left(-\frac{2\pi t}{\sqrt{p}}\right). \]
where $C_k(t)$ is defined by (0.6).

(3) When $t < 0$, $W_{t,\infty}^*(0) = 0$, and

$$W_{t,\infty}^{*,'}(0) = -2ie^{-\pi t/\sqrt{p}}\beta_k(2\pi|t|/\sqrt{p}),$$

where $\beta_k(a)$ is defined by (0.7).

The above two propositions can be proved by standard calculations using lemma 2.1 (see [KRY, Proposition 2.6] for example) and are left to the reader.

Recall ([Ya1]) that $\Phi_q$ is the standard section in $I(s, \chi_q)$ associated to $\text{char}(\mathcal{O}_{E_q}) \in S(E_q)$ with respect to certain Weil representation of $U(1, 1) \times U(1)$ when $q \nmid 2p$.

When $q \mid 2p$, $i(g) \notin H(\mathbb{Z}_p)$ for $g \in G(\mathbb{Q}_q)$, and this prevent $\Phi_q$ from being a standard section.

**Proposition 2.4.** $W_{t,2}^*(s) = 0$ unless $t \in \mathbb{Z}_2$. Assume $t \in \mathbb{Z}_2$, and set $X = 2^{-2s}$.

(1)

$$W_{t,2}^*(s) = \begin{cases} 
-1 & \text{if } t \in \mathbb{Z}_2^*, \\
\sum_{k=0}^{\text{ord}_2 t} (-X)^k & \text{if } t \in 2\mathbb{Z}_2.
\end{cases}$$

In particular,

$$W_{0,2}^*(s) = L(2s, \epsilon_2).$$

(2) When $0 \neq t \in 2\mathbb{Z}_2$, $W_{t,2}^*(0)$ is 1 or 0 depending on whether $\text{ord}_2 t$ is even or odd. Moreover, when $\text{ord}_2 t$ is odd, one has

$$W_{t,2}^{*,'}(0) = (\text{ord}_2 t + 1) \log 2.$$

**Proof.** First notice that when $b \in \frac{1}{2}\mathbb{Z}_2^*$, one has

$$b + \frac{1}{2\delta} = \frac{1 + 2b\delta}{2\delta} \in \mathbb{Q}_2^*.$$

So lemma 2.1 implies

$$W_{t,2}(s) = \int_{\mathbb{Z}_2} \chi(2\delta)|2|_2^{2s+1}\psi(-tb)db + \int_{\frac{1}{2}\mathbb{Z}_2^*} \psi(-tb)db$$

$$+ \sum_{k=2}^{\infty} (-X)^k \int_{\mathbb{Z}_2^*} \psi(-2^{-k}tb)db$$

$$= -\frac{1}{2}X \text{char}({\mathbb{Z}_2})(t) + 2 \text{char}(2\mathbb{Z}_2)(t) - \text{char}({\mathbb{Z}_2})(t)$$

$$+ \sum_{k=2}^{\infty} (-X)^k(\text{char}(2^k\mathbb{Z}_2) - \frac{1}{2} \text{char}(2^{k-1}\mathbb{Z}_2))(t).$$
So \( W_{t,2}(s) = 0 \) unless \( t \in \mathbb{Z}_2 \). When \( t \in \mathbb{Z}_2^* \),
\[
W_{t,2}(s) = -\frac{1}{2}X - 1 = -L(2s + 1, \epsilon_2)^{-1}
\]
as expected. When \( r = \text{ord}_2 t > 0 \),
\[
W_{t,2}(s) = \frac{1}{2}(-X) + 1 + \frac{1}{2} \sum_{k=2}^{r} (-X)^k - \frac{1}{2}(-X)^{r+1}
\]
\[
= (1 + \frac{1}{2}X) \sum_{k=0}^{r} (-X)^k.
\]
So
\[
W^*_{t,2}(s) = \sum_{k=0}^{r} (-X)^k.
\]
The rest follows easily.

**Proposition 2.5.** \( W^*_{t,p}(s) = 0 \) unless \( t \in \mathbb{Z}_p \). Assume \( t \in \mathbb{Z}_p \) and set \( X = p^{-s} \).

1. \[
W^*_{t,p}(s) = \frac{-i}{\sqrt{p}} \left( X - \epsilon_p(t) X^2 \right)^{\text{ord}_p(t+1)}
\]for \( t \neq 0 \), and
\[
W^*_{0,p}(s) = \frac{-i}{\sqrt{p}} X.
\]
2. When \( 0 \neq t \in \mathbb{Z}_p \),
\[
W^*_{t,p}(0) = \begin{cases} 
-\frac{2i}{\sqrt{p}} & \text{if } \epsilon_p(t) = -1, \\
0 & \text{if } \epsilon_p(t) = 1.
\end{cases}
\]
Moreover, when \( \epsilon_p(t) = 1 \),
\[
W^*_{t,p}'(0) = \frac{-2i}{\sqrt{p}} \left( \text{ord}_p t + \frac{1}{2} \right) \log p.
\]

**Proof.** Recall that \( \delta = \sqrt{-p} \) is a uniformizer of \( E_p \). Simple calculation using [Ya3, Lemma 2.4] gives
\[
W_{t,p}(s)
\]
\[
= \int_{\mathbb{Z}_p} (\chi \bar{\eta})(2\delta)|2\delta|^{s+\frac{1}{2}} \psi(-tb) db + \sum_{k=1}^{\infty} X^{2k} \int_{\mathbb{Z}_p^*} \epsilon(b) \psi(-p^{-k}tb) db
\]
\[
= -ip^{-\frac{1}{2}} X \text{ char}(\mathbb{Z}_p)(t) + \sum_{k=1}^{\infty} X^{2k} p^{-\frac{1}{2}} \epsilon_p(-p^k t) \epsilon(\frac{1}{2}, \epsilon_p, \psi_p) \text{ char}(p^{k-1} \mathbb{Z}_p^*)(t).
\]
So \( W_{t,p}(s) = 0 \) unless \( t \in \mathbb{Z}_p \). When \( r = \text{ord}_p t \geq 0 \), recall that \( \epsilon_p(-p^k) = -1 \) and

\[
\epsilon\left(\frac{1}{2}, \epsilon_p, \psi_p\right) = \epsilon\left(\frac{1}{2}, \epsilon_\infty, \psi_\infty\right)^{-1} = -i.
\]

So one has

\[
W_{t,p}^*(s) = W_{t,p}(s) = \frac{-i}{\sqrt{p}}(X - \epsilon_p(t)X^{2(r+1)})
\]
as stated. The other claims now follow easily.

Now we assume that \( q \) is split in \( E \). We make identifications as in [Ya1, section 1.2]. In particular, \( G_q \) is identified with \( \mathbb{Q}^*_q \) via \((z, z^{-1}) \mapsto z\), and \( H_q \) is identified with \( GL_2(\mathbb{Q}_q) \). Also \( \chi_q \) is identified with \( \chi_w = \chi_\infty^{-1} \), and is viewed as a character of \( \mathbb{Q}^*_q \). With respect to the standard basis specified in [Ya1, section 1.2], one has

\[
(2.1) \quad i(g) = \frac{1}{2} \left( \begin{array}{cc} g + 1 & \frac{1}{2x_q}(g - 1) \\ 2x_q(g - 1) & g + 1 \end{array} \right).
\]

Here we have written \( \delta = (x_q, -x_q) \in E_q = \mathbb{Q}^2_q \) with \( x_q \in \mathbb{Z}^*_q \).

**Proposition 2.6.** Let \( g \in \mathbb{Q}^*_q \), and let \( n = |\text{ord}_q g| \). Let \( r = \text{ord}_q t \) and set \( X = q^{-s} \). Then \( W_{t,q}^*(i(g), s) = 0 \) unless \( r \geq -n \). In such a case,

\[
W_{t,q}^*(i(g), s) = \chi_q(g)q^{-\frac{r+n}{2}}\psi_q(\mp \frac{t}{2x_q}) \sum_{k=0}^{r+n} X^{2k-n},
\]

where \( \mp \) depends on whether \( g \in \mathbb{Z}_q \) or not. In particular,

\[
W_{0,q}^*(i(g), s) = \chi_q(g)q^{-\frac{n}{2}}X^{-n}L(2s, \epsilon_q).
\]

**Proof.** Write (for \( b \neq \pm \frac{1}{2x_q} \))

\[
(2.2) \quad wn(b) = n(x)m(y_1, y_2)i(g_0)
\]

with \( x \in \mathbb{Q}_q \), \( y_i \in \mathbb{Q}^*_q \), and \( g_0 \in G(\mathbb{Q}_q) = \mathbb{Q}^*_q \). Here

\[
m(y_1, y_2) = \left( \begin{array}{cc} y_1 & 0 \\ 0 & y_2 \end{array} \right).
\]

Then \( i(g_0) \) fixes \( \pm \frac{1}{2x_q} \)

\[
y_1 = (b + \frac{1}{2x_q})^{-1},
\]

\[
y_2 = b - \frac{1}{2x_q},
\]

\[
g_0 = (y_1y_2)^{-1} = \frac{b + \frac{1}{2x_q}}{b - \frac{1}{2x_q}}.
\]
It is easy to check that $b \mapsto g_b$ is a bijection of $\mathbb{P}^1(\mathbb{Q}_q)$ onto itself. Moreover, for $n \geq 0$,

$$g_b \in q^{-n}\mathbb{Z}_q \text{ if and only if } b \notin \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q,$$

and for $n > 0$,

$$g_b \in q^n\mathbb{Z}_q \text{ if and only if } b \in -\frac{1}{2x_q} + q^n\mathbb{Z}_q.$$

So

$$\Phi_q(wn(b)i(g)) = \chi_q(y_1y_2)|y_1/y_2|_q^{s+\frac{1}{2}}\Phi_q(i(g_b), s)$$

$$= \chi_q^{-1}(g_b)|y_1/y_2|_q^{s+\frac{1}{2}}\Phi_q(i(g_b), s).$$

When $g \in \mathbb{Z}_q$, $n = \text{ord}_q g$. So (2.4) implies that $g_b \in \mathbb{Z}_q$ if and only if $b \notin \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q$. One has then by (1.9)

$$\Phi_q(wn(b)i(g), s) = \chi_q(g)|g|^{-s-\frac{1}{2}}\begin{cases} 1 & \text{if } b \in \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q, \\ |b - \frac{1}{2x_q}|^{-2s-1}|g|^{2s+1} & \text{if } b \notin \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q. \end{cases}$$

Therefore,

$$W_{t, q}(i(g), s)\chi_q^{-1}(g)|g|^{s+1/2}\psi\left(\frac{t}{2x_q}\right)$$

$$= \psi\left(\frac{t}{2x_q}\right)\left(\int_{\frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q} \psi(-tb)db + |g|^{2s+1}\int_{b \notin \frac{1}{2x_q} + q^{n+1}\mathbb{Z}_q} |b - \frac{1}{2x_q}|^{-2s-1}\psi(-tb)db\right)$$

$$= q^{-n-1}\text{ char}(q^{-n-1}\mathbb{Z}_q)(t) + X^{2n}q^{-n}\sum_{k=-n}^\infty X^{2k}\int_{\mathbb{Z}_q^*} \psi(-tb)db$$

$$= q^{-n}\left[\frac{1}{q}\text{ char}(q^{-n-1}\mathbb{Z}_q) + X^{2n}\sum_{k=-n}^\infty X^{2k}\left(\text{ char}(q^k\mathbb{Z}_q) - \frac{1}{q}\text{ char}(q^{k-1}\mathbb{Z}_q)\right)\right](t).$$

So $W_{t, q}(i(g), s) = 0$ unless $r = \text{ord}_q t \geq -n - 1$. When $r = -n - 1$, one has

$$W_{t, q}(i(g), s)\chi_q^{-1}(g)|g|^{s+1/2}\psi\left(\frac{t}{2x_q}\right) = q^{-n}(q^{-1} - X^{2n}q^{-1}X^{-2n}) = 0.$$

When $r \geq -n$, one has

$$W_{t, q}(i(g), s)\chi_q^{-1}(g)|g|^{s+1/2}\psi\left(\frac{t}{2x_q}\right)$$

$$= q^{-n}\left[q^{-1} + (1 - q^{-1})\sum_{k=0}^{r+n} X^{2k} - q^{-1}X^{2r+2n+2}\right]$$

$$= q^{-n}L(2s + 1, \epsilon_q)\sum_{k=0}^{r+n} X^{2k}.$$
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So

\[ W_{t,q}^*(i(g), s) = \chi(g)q^{-\frac{a}{2}} \psi(-\frac{t}{2x_q})X^{-n} \sum_{k=0}^{r+n} X^{2k} \]

as desired. The case \( g \notin \mathbb{Z}_q \) can be proved similarly using (2.5), and is left to the reader.

3. Proof of Theorem 0.1.

We start by choosing \( g_C \) more explicitly. For an ideal class \( C \) of \( E \), choose a primitive ideal \( \mathfrak{a} \in C \) relatively prime to \( 2p \). Write

\[ \mathfrak{a} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r} \]

where \( \mathfrak{P}_i \) are prime ideals of \( E \). Then \( p_i = N\mathfrak{P}_i \) are prime numbers split in \( E \). Recall the lattice decomposition (0.3) of \( \mathfrak{a} \). Then \( a = \prod p_i^{e_i} \). Let \( w_i \) be the place of \( E \) corresponding to \( \mathfrak{P}_i \) and let \( \bar{w}_i \) be the place corresponding to \( \bar{\mathfrak{P}}_i \).

Write \( \delta = (x_i, -x_i) \in E_{p_i} = E_{w_i} \oplus E_{\bar{w}_i} \). Then

\[(3.1) \quad b \equiv -x_i \mod p_i^{e_i} \]

Since

\[ \text{ord}_{w_i}(b + x_i) = \text{ord}_{w_i}(b + \delta) \geq \text{ord}_{\mathfrak{P}_i} \mathfrak{a} = e_i. \]

Define \( a_C = (a_w) \in E_k^* \) such that \( a_w = p_i^{e_i} \) when \( w = w_i \) and \( a_w = 1 \) otherwise. Then the associated ideal of \( a_C \) is \( \mathfrak{a} \). Let \( g_C = (g_q) \) to be the image of \( a_C \) in \( E_k^1 \).

As in [KRY], we define \( \rho_q(n) \), for a prime number \( q \) and an integer \( n \), to be \( \rho(q^{-\text{ord}_q n}) \). We also define \( \rho_{\infty}(n) \) to be 1 or 0 depending on whether \( n \) is positive or not. Finally we define \( \rho(n) = 0 \) when \( n < 0 \). Then \( \rho_p(n) = 1 \), and

\[(3.2) \quad \rho(n) = \prod_q \rho_q(n). \]

**Proposition 3.1.** (1) Assume that \( q \nmid a \) is split. Then \( W_{t,q}^*(i(g_C), s) = 0 \) unless \( t \in \mathbb{Z}_q \). Moreover,

\[ W_{t,q}^*(i(g_C), 0) = (\text{ord}_q t + 1) \text{ char}(\mathbb{Z}_q)(t). \]

(2) When \( q = p_i|a \), \( W_{t,q}^*(i(g_C), s) = 0 \) unless \( t \in p_i^{-e_i} \mathbb{Z}_{p_i} \). In such a case,

\[ W_{t,q}^*(i(g_C), 0) = \frac{\psi_{p_i}(-\frac{t}{\mathfrak{P}_i^{e_i}})\chi(\mathfrak{P}_i^{e_i})}{N\mathfrak{P}_i^{e_i/2}} (\text{ord}_{p_i} t + e_i + 1) \text{ char}(p_i^{-e_i} \mathbb{Z}_{p_i})(t). \]
(3) For \(0 \neq t \in \mathbb{Z}\), one has
\[
\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W_{\frac{t}{a}, q}^*(i(g_C), 0) = \frac{\mu(\mathfrak{A})}{a^{k+1}} (-1)^t e^{\frac{\pi itb}{ap}} \prod_{q \in S_{sp}} \rho_q(t).
\]

(4) For \(t = 0\), one has
\[
\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W_{0, q}^*(i(g_C), 0) = \mu(\mathfrak{A}) \prod_{q \in S_{sp}} \rho_q(t) \prod_{i} \psi_{p_i}(-\frac{t}{2x_i a}).
\]

Proof. Claims (1), (2), and (4) follow from Proposition 2.6 immediately. For (3), first notice that for an integer \(t \neq 0\),
\[
(3.3) \quad \rho_q(t) = \text{ord}_q t/a + 1 = \begin{cases} \text{ord}_q t/a + 1 & \text{if } q \nmid a, \\ \text{ord}_q t/a + e_i + 1 & \text{if } q = p_i|a. \end{cases}
\]
So (1) and (2) imply (recall that \(\mu = \chi_{\tilde{\eta}}|_{\mathbb{A}}^{k-\frac{1}{2}}\))
\[
\tilde{\eta}(\mathfrak{A}) \prod_{q \in S_{sp}} W_{t/a, q}^*(i(g_C), 0) = \frac{\mu(\mathfrak{A})}{a^{k+1}} \prod_{q \in S_{sp}} \rho_q(t) \prod_{i} \psi_{p_i}(-\frac{t}{2x_i a}).
\]
Choose integers \(\tilde{2}, \tilde{x}_i, \text{ and } \tilde{a}_i\) such that
\[
2\tilde{2} \equiv 1 \mod pa,
\]
and
\[
x_i \tilde{x}_i \equiv 1 \mod p_i^{e_i}, \quad a_i \tilde{a}_i \equiv 1 \mod p_i^{e_i}.
\]
Here \(a_i = ap^{-e_i}\). Then
\[
\psi_{p_i}(-\frac{t}{2x_i a}) = e\left(\frac{t2\tilde{x}_i \tilde{a}_i}{p_i^{e_i}}\right),
\]
and
\[
\prod_{i} \psi_{p_i}(-\frac{t}{2x_i a}) = e\left(\frac{t2\tilde{b}}{a}\right).
\]
Here \(e(z) = e^{2\pi i z}\), and \(\tilde{b} = \sum_i \tilde{x}_i a_i \tilde{a}_i \equiv \tilde{x}_i \mod p_i^{e_i}\). Since \(b \equiv -x_i \mod p_i^{e_i}\) for all \(i\), and \(b^2 \equiv -p \mod 4ap\), one has
\[
b\tilde{b} \equiv -1 \mod a,
\]
and
\[
p\tilde{b} \equiv -b^2 \tilde{b} \equiv b \mod ap.
\]
So
\[
\prod_{i} \psi_{p_i}(-\frac{t}{2x_i a}) = e\left(\frac{tb\tilde{2}}{ap}\right) = e^{\pi itb \frac{2\tilde{2}}{ap}} = (-1)^t e^{\frac{\pi itb}{ap}} = (-1)^t e^{\frac{\pi itb}{ap}}.
\]
This proves (3).

The following lemma follows from propositions 2.2-2.5 and 3.1 immediately.
Lemma 3.2. $E_t^*(i(g_C), s) = 0$ unless $t \in \frac{1}{a}\mathbb{Z}$.

Define
\begin{equation}
\Lambda(s, \mu, C) = \frac{1}{2} \tilde{\eta}(A) E^*(i(g_C), s)
\end{equation}
and
\begin{equation}
\Lambda_t(s, \mu, C) = \frac{1}{2} \tilde{\eta}(A) E^*_t(i(g_C), s).
\end{equation}
Then $\Lambda_t(s, \mu, C) = 0$ unless $t$ is an integer by Lemma 3.2, and
\begin{equation}
L'(k + 1, \mu) = \pi \sum_C \Lambda'(0, \mu, C)
\end{equation}
by proposition 1.1.

Lemma 3.3. One has
\begin{equation}
\Lambda'(0, \mu, C) = \sum_{q \leq \infty} \sum_{\text{inert } t \in q\mathcal{N}E^*} \Lambda'_t(0, \mu, C) + \sum_{t \in \mathcal{N}E^*} \Lambda'_t(0, \mu, C) + \sum_{t \in -\mathcal{N}E^*} \Lambda'_t(0, \mu, C) + \Lambda'_0(0, \mu, C).
\end{equation}
Here the sums are over integers $t$.

Proof. This is a special case of a general principle discovered by Kudla ([Ku]). The key point is the following fact: In a nondegenerate Fourier coefficient of the incoherent Eisenstein series, at least one local Whittaker function vanishes at the center, and whether it vanishes is controlled by local root numbers, the local epsilon dichotomy principle ([HKS]). We gave a direct proof of this lemma here to show the general principle. For each rational integer $t \neq 0$, let $D(t)$ be the set of primes $q$ not satisfying the local epsilon dichotomy condition
\begin{equation}
\epsilon(\frac{1}{2}, (\chi \tilde{\eta})_q, \frac{1}{2} \psi_{E_q})(\chi \tilde{\eta})_q(\delta) = \epsilon_q(t).
\end{equation}
Here $\psi = \prod \psi_q$ is the ‘canonical’ nontrivial additive character of $\mathbb{Q}_\mathfrak{a}/\mathbb{Q}$ as in [Ya2] and $\frac{1}{2} \psi_{E_q}(z) = \psi(\frac{1}{2} \text{tr}_{E_q/\mathbb{Q}} z)$, and $\epsilon(\frac{1}{2}, (\chi \tilde{\eta})_q, \frac{1}{2} \psi_{E_q})$ is Tate’s local root number. More concretely, (1) : $p \in D(t)$ if and only if $\epsilon_p(t) = 1$, (2): For a prime $q \neq p$, $q \in D(t)$ if and only if $\epsilon_q(t) = -1$. It is easy to see that $D(t)$ is a finite set of nonsplit primes of odd order. Notice that $D(t) = D(\frac{1}{a})$ since $\mathfrak{a}$ is a norm from $E^*$. The key is that $W^*_{\mathfrak{a}, q}(0) = 0$ for every $q \in D(t)$ (which is clear from calculations in section 2). So $E^*_t(i(g_C), 0) = 0$ unless $|D(t)| = 1$. For each
prime \( q \), collect the terms \( \Lambda'_t(0) \) together for all \( t \) such that \( D(t) = \{ q \} \). Since the set
\[
\{ t \neq 0 : D(t) = \{ q \} \} = \begin{cases} 
NE^* & \text{if } q = p, \\
-NE^* & \text{if } q = \infty, \\
qNE^* & \text{if } (\frac{q}{p}) = -1,
\end{cases}
\]
one proves the lemma.

**Proposition 3.4.** Let \( t \) be a nonzero integer and let \( e_p(z) = e^{\frac{2\pi i z}{p}} \). Then
\[
\Lambda'_t(0, \mu, C) = -\frac{2}{\sqrt{p}} \frac{\mu(\mathfrak{a})}{a^{k+1}} e_p(t \tau_{\mathfrak{a}}) \left\{ \begin{array}{l}
\rho(t) ( \text{ord}_t t + 1) C_k(\frac{-2\pi t}{a\sqrt{p}}) \log q & \text{if } \rho(t) \neq 0, \\
\rho(t) ( \text{ord}_p t + \frac{1}{2}) C_k(\frac{-2\pi t}{a\sqrt{p}}) \log p & \text{if } \rho(t) = 0, \\
\rho(-t) \beta_k(\frac{-2\pi t}{a\sqrt{p}}) & \text{if } \rho(-t) = 0, \\
0 & \text{otherwise.}
\end{array} \right.
\]

Here we require \( q \) to be inert when \( \rho(t/q) \neq 0 \).

**Proof.** By propositions 2.2 and 2.4, one has
\[
W^*_t, q(0) = \left\{ \begin{array}{l}
\rho(t) & \text{if } q \nmid 2\infty \text{ is inert}, \\
(-1)^t \rho_2(t) & \text{if } q = 2.
\end{array} \right.
\]

Assume first \( \rho(t/q) \neq 0 \) for an inert prime \( q \). Then \( q \) is unique and \( W^*_n(0) = 0 \). In this case, \( t > 0 \) and \( e_p(\frac{t}{a}) = e_p(q) = -1 \). So
\[
W^*_n, \infty(0) = -2i \rho_\infty(t) C_k(\frac{-2\pi t}{\sqrt{a}}) e^{\frac{-\pi t}{a\sqrt{p}}}
\]
and
\[
W^*_n, p(0) = -\frac{2i}{\sqrt{p}} \rho_p(t)
\]
by propositions 2.3 and 2.5. Also \( \prod_{l \neq q} \rho_l(t) = \rho(t/q) \) in this case. So one has by propositions 2.2, 2.4 and 3.1 and formulae (3.7)–(3.9)
\[
\Lambda'_t(0, \mu, C) = \frac{1}{2} \tilde{\eta}(\mathfrak{a}) E_{\frac{t}{a}}^*(i(g_C), 0)
\]
\[
= \frac{1}{2} W^*_n, q(0) \prod_{l \neq q, \text{nonsplit}} W^*_n, l(0) \cdot \tilde{\eta}(\mathfrak{a}) \prod_{l \text{ split}} W^*_n, l(i(g_C), 0)
\]
\[
= \frac{1}{2} ( \text{ord}_q t + 1) \log q(-1)^t \frac{-2i}{\sqrt{p}} (-2i) C_k(\frac{-2\pi t}{a\sqrt{p}}) e^{\frac{-\pi t\sqrt{p}}{a p}} \prod_{l \neq q} \rho_l(t) \frac{\mu(\mathfrak{a})}{a^{k+1}} (-1)^t e^{\frac{\pi t l h}{a p}}
\]
\[
= -\frac{2}{\sqrt{p}} \rho(t/q) ( \text{ord}_q t + 1) \log q \cdot \frac{\mu(\mathfrak{a})}{a^{k+1}} C_k(\frac{-2\pi t}{a\sqrt{p}}) e_p(t \tau_{\mathfrak{a}})
\]
as stated.

Next, assume \( \rho(t) \neq 0 \), i.e., \( t \) is a norm from \( E^* \). In this case, \( W_{\frac{n}{p}}^{*}(0) = 0 \).

By propositions 2.5, 3.1 and formulae (3.7) – (3.8), one has then

\[
\Lambda'(0, \mu, C) = \frac{1}{2} \tilde{\eta}(\mathfrak{A}) E_{\frac{n}{p}}^{*'}(i(g_C), 0)
\]

\[
= \frac{1}{2} W_{\frac{n}{p}}^{*'}(0) \prod_{l \neq p, \text{ inert}} W_{\frac{n}{p},l}^{*}(0) \cdot \tilde{\eta}(\mathfrak{A}) \prod_{l \text{ split}} W_{\frac{n}{p},l}^{*}(i(g_C), 0)
\]

\[
= \frac{1}{2} \frac{-2i}{\sqrt{p}} \left( \text{ord}_p t + \frac{1}{2} \right) \log p (-1)^t (-2iC_k\left(-\frac{2\pi t}{a\sqrt{p}}\right))e^{-\frac{\pi t}{\sqrt{p}}} \prod_{l \neq p} \rho_l(t) \mu(\mathfrak{A}) \frac{a}{a^k+1} (-1)^t e^{\frac{\pi itb}{ap}}
\]

\[
= -\frac{2}{\sqrt{p}} \rho(t) \left( \text{ord}_p t + \frac{1}{2} \right) \log p \cdot \frac{\mu(\mathfrak{A})}{a^k+1} C_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e_p(t\tau_{\mathfrak{A}})
\]

as stated.

Finally, assume \( \rho(-t) \neq 0 \), i.e., \( -t \) is a norm. In this case \( W_{\frac{n}{p},\infty}^{*}(0) = 0 \). Also \( \epsilon_p(\frac{n}{p}) = \epsilon_p(-1) = -1 \), so \( W_{l,p}^{*} \) is given by (3.9). By propositions 2.3, 2.5, and formulae (3.7) – (3.9), one has

\[
\Lambda'(0, \mu, C) = \frac{1}{2} \tilde{\eta}(\mathfrak{A}) E_{\frac{n}{p}}^{*'}(i(g_C), 0)
\]

\[
= \frac{1}{2} W_{\frac{n}{p},\infty}^{*'}(0) \prod_{l<\infty, \text{ nonsplit}} W_{\frac{n}{p},l}^{*}(0) \cdot \tilde{\eta}(\mathfrak{A}) \prod_{l \text{ split}} W_{\frac{n}{p},l}^{*}(i(g_C), 0)
\]

\[
= \frac{1}{2} (-2i) \beta_k\left(-\frac{2\pi t}{a\sqrt{p}}\right)e^{-\frac{\pi t}{\sqrt{p}}} \prod_{l<\infty} \rho_l(t) \mu(\mathfrak{A}) \frac{a}{a^k+1} (-1)^t e^{\frac{\pi itb}{ap}}
\]

\[
= -\frac{2}{\sqrt{p}} \rho(-t) \beta_k\left(-\frac{2\pi t}{a\sqrt{p}}\right) \mu(\mathfrak{A}) \frac{a}{a^k+1} e_p(t\tau_{\mathfrak{A}})
\]

as stated.

**Proposition 3.5.** For the constant term, one has

\[
\Lambda_0(s, \mu, C) = \frac{2^s \mu(\mathfrak{A})}{2a^{k+1}p^s \prod_{j=1}^k (j+s)} \left[ G\left(\frac{p}{2a}, s\right) - G\left(\frac{p}{2a}, -s\right)\right]
\]

where

\[
G(y, s) = y^s \Lambda(1+2s, \epsilon) \prod_{J=1}^k (j+s).
\]

Moreover,

\[
\Lambda'(0, \mu, C) = \frac{h_p \mu(\mathfrak{A})}{\sqrt{pa^{k+1}}} \left[ 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)} + \log \frac{p}{2a} + \sum_{j=1}^k \frac{1}{j} \right].
\]
Proof. First one has by definition $\Phi_q(i(g_C), s) = 1$ unless $q = p_i$ for some $i$. In that case

$$\Phi_{p_i}(i(g_C), s) = \chi_{p_i}(p_i^{e_i}) p^{-e_i(s + \frac{1}{2})} = \chi(p_i^{e_i})(N p_i^{e_i})^{-\frac{1}{2}}(N p_i)^{-e_is}.$$ 

So

$$\tilde{\eta}(\mathfrak{A}) \Phi^*(i(g_C), s) = \tilde{\eta}(\mathfrak{A}) \Lambda(2s + 1, \epsilon) \prod \Phi_q(i(g_C), s) = \frac{\mu(\mathfrak{A})}{a^{k+1}} a^{-s} \Lambda(2s + 1, \epsilon).$$

On the other hand, one has by propositions 2.2-2.5 and 3.1

$$\tilde{\eta}(\mathfrak{A}) \prod_{q \leq \infty} W_{0,q}^*(i(g_C), s) = \Lambda(2s, \epsilon) (-i2^{2s}p^s) \frac{-i}{\sqrt{p}} p^{-\frac{1}{2}} a^s \prod_{j=1}^k \frac{j - s}{j + s}$$

$$= -2^{2s} a^s p^{-\frac{1}{2}} \mu(\mathfrak{A}) a^{k+1} \Lambda(2s, \epsilon) \prod_{j=1}^k \frac{j - s}{j + s}.$$ 

So

$$\Lambda_0(s, \mu, C) = \frac{1}{2} \tilde{\eta}(\mathfrak{A}) \left[ \Phi^*(i(g_C), s) + \prod_{q \leq \infty} W_{0,q}^*(i(g_C), s) \right]$$

$$= \frac{\mu(\mathfrak{A})}{2a^{k+1}} [a^{-s} \Lambda(2s + 1, \epsilon) - 2^{2s} a^s p^{-\frac{1}{2}} \Lambda(2s, \epsilon) \prod_{j=1}^k \frac{j - s}{j + s}]$$

The functional equation

$$p^{\frac{1+s}{2}} \Lambda(s, \epsilon) = p^{\frac{2-s}{2}} \Lambda(1-s, \epsilon),$$

implies

$$p^{-\frac{1}{2}} \Lambda(2s, \epsilon) = p^{-2s} \Lambda(1-2s, \epsilon).$$

So

$$\Lambda_0(s, \mu, C) = \frac{2^s \mu(\mathfrak{A})}{2a^{k+1} p^s \prod_{j=1}^k (j + s)} \left( G\left( \frac{p}{2a}, s \right) - G\left( \frac{p}{2a}, -s \right) \right)$$

as desired. It is obvious from this formula

$$\Lambda'_0(0, \mu, C) = \frac{\mu(\mathfrak{A}) G\left( \frac{p}{2a}, 0 \right) G'\left( \frac{p}{2a}, 0 \right)}{a^{k+1} k!} \left( G\left( \frac{p}{2a}, 0 \right) - G\left( \frac{p}{2a}, 0 \right) \right)$$

$$= \frac{\mu(\mathfrak{A}) \Lambda(1, \epsilon)}{a^{k+1}} \left[ \log \frac{p}{2a} + 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)} + \sum_{j=1}^k \frac{1}{j} \right]$$
Now the final formula follows from the well known number theory fact

\[(3.10) \quad \Lambda(1, \epsilon) = \frac{h_p}{\sqrt{p}}.\]

**Proof of Theorem 0.1** By Propositions 3.4 and 3.5, we have

\[
\Lambda'(0, \mu, C) = \frac{\mu(2\mathfrak{A})}{\sqrt{p}a^{k+1}} \left[ \frac{h_p}{2a} \left( \frac{p}{2a} \right) + 2\frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)} + \sum_{j=1}^{k} \frac{1}{j} \right] \\
- \frac{2}{\sqrt{p}} \sum_{n>0} a_n C_k(-\frac{2\pi n}{a\sqrt{p}}) e_p(n\tau_{2\mathfrak{A}}) \\
- \frac{2}{\sqrt{p}} \sum_{n<0} \rho(-n) \beta_k(-\frac{2\pi n}{a\sqrt{p}}) e_p(n\tau_{2\mathfrak{A}}) \right].
\]

It is clear from this formula that

\[\Lambda'(0, \mu \xi, C) = \xi(C)\Lambda'(0, \mu, C)\]

for every ideal class character \(\xi\). Combining this with (0.2) and (3.6), one sees that

\[(3.11) \quad L'(k+1, \mu, C) = \pi \Lambda'(0, \mu, C).\]

This proves the main formula.

**Remark 3.6.** It is easy to see by 3.1 that

\[\Lambda(s, \mu \xi, C) = \xi(C)\Lambda(s, \mu, C)\]

for every ideal class character \(\xi\) of \(E\). Combining this with (1.10), we see that

\[\Lambda(s, \mu, C) / L(2s+1, \epsilon_{\infty})\]

satisfies (0.2) with a proper shift on \(s\). So

\[(3.12) \quad \Lambda(s, \mu, C) = L(2s+1, \epsilon_{\infty})L(s+k+1, \mu, C).\]

**4. A variant.**

Define two functions

\[(4.1) \quad \theta_k(\tau) = h_p + 2 \sum_{n>0} \rho(n) C_k(-4\pi n \ \text{Im}(\tau)) e(n\tau)\]
and
\[(4.2) \quad \phi_k(\tau) = a_0(\tau) - 2 \sum_{n > 0} a_n C_k(-4\pi n \text{Im} (\tau))e(n\tau) - 2 \sum_{n < 0} \rho(-n)\beta_k(-4\pi n \text{Im} (\tau))e(n\tau).\]

Here
\[(4.3) \quad a_0(\tau) = h_p \left( \frac{3}{2} \log p + \log \text{Im} (\tau) + 2 \frac{A'(1, \epsilon)}{A(1, \epsilon)} + \sum_{j=1}^{k} \frac{1}{j} \right).\]

Both functions are closely related to the Taylor expansion of the well known Eisenstein series
\[E_k(\tau, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(q)} \epsilon(d)(c\tau + d)^{-2k-1} |c\tau + d|^{-2s}.\]
at the symmetric center ([Ya4]) . When \(k = 0\), \(\phi_0\) is the modular form constructed in [KRY] (up to a sign).

First we notice that theorem 0.1 can be rewritten as
\[(4.4) \quad L'(k+1, \mu, C) = \frac{\pi \mu(\mathfrak{p})}{\sqrt{pN(\mathfrak{p})^{k+1}}} \phi_k(\frac{\tau_{\mathfrak{p}}}{p})\]
for every ideal class \(C\) of \(E = \mathbb{Q}(\sqrt{-p})\). In particular,
\[(4.5) \quad L'(k+1, \mu, \text{trivial}) = \frac{\pi}{\sqrt{p}} \phi_k\left( \frac{1}{2} + \frac{1}{2\sqrt{p}i} \right).\]

This is the first formula in theorem 0.2 when \(k = 0\). The purpose of this section is to prove

**Theorem 4.1.** Let the notation be as above. Then
\[L'(k+1, \mu, \text{trivial}) = \frac{4\pi}{\sqrt{p}} \left( \phi_k\left( \frac{2i}{\sqrt{p}} \right) - \theta_k\left( \frac{i}{\sqrt{p}} \right) \log 2 \right).\]

That is
\[\frac{\sqrt{p}}{4\pi} L'(k+1, \mu, \text{trivial}) = h_p(\log p + 2 \frac{A'(1, \epsilon)}{A(1, \epsilon)} + \sum_{j=1}^{k} \frac{1}{j})\]
\[\quad - 2 \sum_{n > 0} a_n C_k(-\frac{8\pi n}{\sqrt{p}})e^{-\frac{4\pi n}{\sqrt{p}}}\]
\[\quad - 2 \sum_{n < 0} \rho(-n)\beta_k(-\frac{8\pi n}{\sqrt{p}})e^{-\frac{4\pi n}{\sqrt{p}}}\]
\[\quad - 2 \log 2 \sum_{n > 0} \rho(n)C_k(-\frac{4\pi n}{\sqrt{p}})e^{-\frac{2\pi n}{\sqrt{p}}}\]

When \(k = 0\), this gives the second formula in theorem 0.2.
Lemma 4.2. One has
\[ \phi_k \left( \frac{1}{2} + \tau \right) = 2\phi_k(4\tau) - \phi_k(\tau) - 4\theta_k(2\tau) \log 2. \]

Proof. Define \( a_n = \rho(n) = 0 \) when \( n \) is not an integer. Since 2 is inert in \( E = \mathbb{Q}(\sqrt{-p}) \), one has for \( n > 0 \)
\[ (-1)^n a_n = -a_n + 2a_{\frac{n}{2}} + 4 \log 2 \rho \left( \frac{n}{2} \right). \]
Indeed, when \( n \) is odd, it is trivial. When \( n \equiv 2 \mod 4 \),
\[ (-1)^n a_n = a_n = 2 \log 2 \rho \left( \frac{n}{2} \right) = a_{\frac{n}{2}} + 2 \log 2 \rho \left( \frac{n}{2} \right). \]
When \( n \equiv 0 \mod 4 \), one has
\[ (-1)^n a_n = a_n = a_{\frac{n}{2}} + 2 \log 2 \rho \left( \frac{n}{8} \right) = a_{\frac{n}{2}} + 2 \log 2 \rho \left( \frac{n}{2} \right). \]
For \( n < 0 \), one has similarly
\[ (-1)^n \rho(-n) = -\rho(-n) + 2\rho \left( -\frac{n}{4} \right). \]
Finally,
\[ a_0(4\tau) = 2 \log 2 + a_0(\tau) = 2 \log 2 + a_0 \left( \frac{1}{2} + \tau \right). \]
Now the lemma follows from a simple calculation.

For \( \tau = x + \frac{y}{2\sqrt{p}} i \in \mathfrak{f} \), the upper half plane, set
\[ g_\tau = n(x)m(\sqrt{y}) \in G(\mathbb{R}). \]
We remark that \( g_\tau \left( -\frac{1}{25} \right) = \tau. \n
Lemma 4.3. Let the notation be as above. Then
\[ \phi_k \left( \frac{1}{2} + \tau \right) = \frac{\sqrt{p}}{2\sqrt{y}} E^*(g_\tau, 0). \]

Proof. (Sketch) A simple calculation gives
\[ W_{d,\infty}^*(g_\tau, s) = y^{\frac{1}{2} - \frac{s}{2}} e(dx) W_{dy,\infty}^*(s). \]
So one has for $d \neq 0$,

$$\mathcal{E}_d^*(g_\tau, s) = \mathcal{E}_d^*(1, s) \frac{W_{dy, \infty}^*(s)}{W_{d, \infty}^*(s)} y^{1/2-s} e(dx),$$

and

$$\mathcal{E}_d^{*,\prime}(g_\tau, 0) = \mathcal{E}_d^{*,\prime}(1, 0) \lim_{s \to 0} \frac{W_{dy, \infty}^*(s)}{W_{d, \infty}^*(s)} y^{1/2} e(dx).$$

Recall that (formula (3.4))

$$\mathcal{E}_d^{*,\prime}(1, 0) = 2\Lambda'(0, \mu, \text{trivial}).$$

Now propositions 2.3 and 3.4 give (the ideal class $C$ is trivial here)

$$\mathcal{E}_d^{*,\prime}(g_\tau, 0) = \begin{cases} 
\frac{4\sqrt{y}}{\sqrt{p}} (-1)^{n-1} a_n C_k(-4\pi n \text{Im } (\tau)) e(n\tau) & \text{if } n > 0, \\
\frac{4\sqrt{y}}{\sqrt{p}} (-1)^{n-1} \rho(-n) \beta_k(-4\pi n \text{Im } (\tau)) e(n\tau) & \text{if } n < 0.
\end{cases}$$

Similar calculation to the proof of proposition 3.5 gives

$$\mathcal{E}_0^*(g_\tau, s) = \frac{2^s y^{1/2}}{p^s \prod_{j=1}^{k} (j+s)} (G\left(\frac{py}{2}, s\right) - G\left(\frac{py}{2}, -s\right)).$$

So

$$\mathcal{E}_0^{*,\prime}(g_\tau, 0) = 2y^{1/2} \Lambda(1, \epsilon)(\log \frac{py}{2} + 2 \frac{\Lambda'(1, \epsilon)}{\Lambda(1, \epsilon)}) + \sum_{j=1}^{k} \frac{1}{j}$$

$$= \frac{2\sqrt{y}}{\sqrt{p}} a_0(\tau).$$

Therefore,

$$\mathcal{E}_0^{*,\prime}(g_\tau, 0) = \frac{2\sqrt{y}}{\sqrt{p}} \phi_k\left(\frac{1}{2} + \tau\right)$$

as desired.

**Proposition 4.4.** One has the functional equation.

$$\phi_k\left(-\frac{1}{p\tau}\right) = -\sqrt{p}|\tau|\left(-\frac{\delta\tau}{|\delta\tau|}\right)^{2k+1} \phi_k(\tau).$$
Proof. (Sketch) By lemma 4.3, this proposition follows from the following trivial functional equation plus a long tedious calculation on both sides.

\[(4.7) \quad E^*(\gamma_f^{-1}g, s) = E^*(\gamma_{\infty}g, s)\]

for any \(\gamma \in G(\mathbb{Q})\). Here \(\gamma_f\) and \(\gamma_{\infty}\) are the finite and infinite part of \(\gamma\), viewed as an element in \(G(\mathbb{A})\). Indeed, taking \(g = g_\tau\) and \(\gamma = \frac{1}{2} \alpha n(-\frac{1}{2})\) with

\[\alpha = \begin{pmatrix} 0 & -\frac{1}{\delta} \\ -\delta & 0 \end{pmatrix},\]

and computing both sides of (4.7), one gets

\[(4.8) \quad -E^*,'(g_\tau, 0) = (\frac{|-\delta(\tau - \frac{1}{2})|}{-\delta(\tau - \frac{1}{2})})^{2k+1}E^*,'(g_{\tau'}, 0),\]

with \(\tau' = \frac{1}{2} - \frac{1}{p(\tau - \frac{1}{2})}\). Replacing \(\tau\) by \(\frac{1}{2} + \tau\), this is exactly what we claimed in this proposition by lemma 4.3. Here is a sketch to derive (4.8) from (4.7). First,

\[\gamma_{\infty}g_\tau = tg_{\tau'}i(-t^2)\]

with \(t = \frac{-\delta(\tau - \frac{1}{2})}{|\delta(\tau - \frac{1}{2})|}\), and so

\[W_{d,\infty}^*(\gamma_{\infty}g_\tau, s) = (\frac{|-\delta(\tau - \frac{1}{2})|}{-\delta(\tau - \frac{1}{2})})^{2k+1}W_{d,\infty}^*(g_{\tau'}, s)\]

and

\[E^*(\gamma_{\infty}g_\tau, s) = (\frac{|-\delta(\tau - \frac{1}{2})|}{-\delta(\tau - \frac{1}{2})})^{2k+1}E^*(g_{\tau'}, s)\]

On the other hand, when \(q\) is non split,

\[\gamma_q^{-1} = n(x)m(y)i(g)\]

with

\[y = \frac{2}{1 - \delta}, \quad g = \frac{1 - \delta}{1 + \delta}, \quad x = \frac{5 + p}{2(1 + p)}\]

This implies

\[W_{d,q}^*(\gamma_q^{-1}, s) = (\chi\tilde{n})q(\frac{2}{1 - \delta})\tilde{n}_q(\delta)W_{d,q}^*(s)\]

When \(q\) is split, \(\delta = (x_q, -x_q)\), and

\[\gamma_q^{-1} = n(x) \text{ diag}(y_1, y_2)i(g)\]
with
\[ y_1 = \frac{2}{1 - x_q}, \quad y_2 = \frac{1 + x_q}{2}, \quad g = -\frac{1 - x_q}{1 + x_q}. \]
So (after a long calculation using results in section 2)
\[ W_{d,q}^* (\gamma_q^{-1}, s) = W_{d,q}^* (s). \]
So one has for \( d \neq 0 \)
\[ E_d^* (g_\tau, s) = \prod_{q \text{ nonsplit}} (\chi_\tilde{\eta}_q)(\frac{2}{1 - \delta})\tilde{\eta}_q(\delta)E_d^* (g_\tau, s) = -E_d^* (g_\tau, s). \]
The same is true for \( d = 0 \). Therefore (4.7) implies
\[ -E^* (g_\tau, s) = (\frac{|\delta(\tau - \frac{1}{2})|}{-\delta(\tau - \frac{1}{2})})^{2k+1}E^* (g_\tau', s). \]
Taking derivative on both sides at \( s = 0 \), one gets (4.8).

**Proof of Theorem 4.1** Now the proof of theorem 4.1 becomes easy. Indeed, taking \( \tau = \frac{1}{2\sqrt{p}}i \) in proposition 4.4, one gets
\[ \phi_k \left( \frac{1}{2\sqrt{p}}i \right) = -2\phi_k \left( \frac{2}{\sqrt{p}}i \right). \]
Now (4.5) and lemma 4.2 imply
\[ \frac{\sqrt{p}}{\pi} L'(k + 1, \mu, \text{trivial}) = \phi_k \left( \frac{1}{2} + \frac{i}{\sqrt{p}} \right) \]
\[ = 2\phi_k \left( \frac{2i}{\sqrt{p}} \right) - \phi_k \left( \frac{i}{2\sqrt{p}} \right) - 4\theta_k \left( \frac{i}{\sqrt{p}} \right) \log 2 \]
\[ = 4\phi_k \left( \frac{2i}{\sqrt{p}} \right) - 4\theta_k \left( \frac{i}{\sqrt{p}} \right) \log 2 \]
So
\[ L'(k + 1, \mu, \text{trivial}) = \frac{4\pi}{\sqrt{p}}\phi_k \left( \frac{2i}{\sqrt{p}} \right) - \theta_k \left( \frac{i}{\sqrt{p}} \right) \log 2 \]
as claimed.

Finally, we would like to mention that S. Miller and the author ([MY]) have just proved that the central derivative \( L'(1, \mu) \neq 0 \) for every canonical Hecke character \( \mu \) of \( \mathbb{Q} (\sqrt{-p}) \) of weight one. This implies the following inequality:
\[ \phi_0 \left( \frac{2i}{\sqrt{p}} \right) > \theta_0 \left( \frac{i}{\sqrt{p}} \right) \log 2, \]
for \( p \equiv 3 \text{ mod } 8 \). It would be interesting to prove this inequality directly, which would give an independent proof of \( L'(1, \mu) \neq 0. \)
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