CHOWLA-SELBERG FORMULA AND COLMEZ’S CONJECTURE

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Abstract. In this paper, we reinterpret the Colmez conjecture on Faltings’ height of CM abelian varieties in terms of Hilbert (and Siegel) modular forms. We construct an elliptic modular form involving Faltings’ height of a CM abelian surface and arithmetic intersection numbers, and prove that Colmez’s conjecture for CM abelian surfaces is equivalent to the cuspidality of this modular form.

1. Introduction

The celebrated Chowla-Selberg formula [CS] asserts:

\[
\prod_{[a] \in \mathcal{O}_L(K)} |\Delta(\tau_a)| \text{Im}(\tau_a)^6 = \left(\frac{1}{2\pi i}\right)^h \prod_{0 < c < L} \Gamma\left(\frac{c}{l}\right)^{6\epsilon(c)}.
\]

Here \(K = \mathbb{Q}(\sqrt{-l})\) is an imaginary field of prime discriminant \(-l\), \(h\) is the ideal class number of \(K\), and \(\epsilon(c) = \left(\frac{c}{l}\right)\). Moreover, \(\Delta\) is the well-known cusp form of weight 12, and \(\Gamma(x)\) is the usual Gamma function. Gross re-interpreted this formula (up to a constant multiple in \(\bar{\mathbb{Q}}\)) as a period relation for a CM elliptic curve in his thesis [Gr1]. Later, he generalized this period relation to an CM abelian variety with CM by a CM abelian extension of \(\mathbb{Q}\) [Gr2]. Anderson reformulate the right-hand-side of Gross’s formula [An] in terms of log-derivative of Dirichlet L-series. In 1993, Colmez [Co] defined \(p\)-adic periods of a CM abelian variety (using integral model) and conjectured that there should be a product formula for periods. Using that, he derived a conjecture which gives a very precise identity between the Faltings’ height of a CM abelian variety and the logarithmic derivative of certain virtual Artin L-function at \(s = 0\). It can be roughly stated as follows. Let \(K\) be a CM number field and let \(\Phi\) be a CM type of \(K\). Let \(A\) be a CM abelian variety of CM type \((\mathcal{O}_K, \Phi)\) defined over a number field \(L\) such that \(A\) has good reduction everywhere, and let \(\alpha \in \Lambda^g\Omega_A\) be a Neron differential of \(A\) over \(\mathcal{O}_L\), non-vanishing everywhere. Then the Faltings height of \(A\) is defined as (our normalization is slightly different from that of [Co]):

\[
h_{\text{Fal}}(A) = \frac{1}{2[L : \mathbb{Q}]} \sum_{\sigma : L \to \mathbb{C}} \log \left( \frac{1}{2\pi i} \right)^g \int_{\sigma(A)(\mathbb{C})} \sigma(\alpha) \wedge \overline{\sigma(\alpha)} + \log \#\Lambda^g\Omega_A/\mathcal{O}_L\alpha.
\]
Here $g = \dim A$. It is independent of the choice of $L$. In fact, Colmez proved that $h_{\text{Fal}}(\Phi) = \frac{1}{|K:Q|} h_{\text{Fal}}(A)$ depends only on the CM type $\Phi$, not on $A$ or $K$ ([Co, Theorem 0.3]). On the other hand, Colmez constructed a class function $A_0^{\Phi}$ on $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ from the CM type $\Phi$ (see Section 3 for detail), which can be viewed as a linear combination of characters of Artin representations, say $\sum a_\chi \chi$. The Colmez conjecture asserts

(1.3) \[ h_{\text{Fal}}(\Phi) = -\sum a_\chi \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{2} \sum a_\chi \log f_{\text{Art}}(\chi) + \frac{1}{4} \log 2\pi \]

where $f_{\text{Art}}(\chi)$ is the analytic Artin conductor of $\chi$.

When the CM abelian variety is an elliptic curve, it is a reformulation of the Chowla-Selberg formula. In the same paper, he proved the conjecture for an abelian CM number field, by combining Gross’s work with his computation of $p$-adic period of the Jacobian of the Fermat curves. A less precise version of the conjecture and the result have been recently generalized to CM motives by V. Maillot and Roessler [MR] and Köhler and Roessler [KR] using Lefschetz fixed point theorem in Arakelov geometry. Yoshida independently developed conjectures about absolute CM period which is very close to Colmez’s conjecture and provided some non-trivial numerical evidence as well as partial results (see for example [Yo]). We should also mention that van der Pooten and Williams [VW] gave another proof of the Chowla-Selberg formula by computing the CM values of the $\eta$-function.

Nothing is known about Colmez’s conjecture besides what he has proved. It remains a mystery in the non-abelian case. The goal of this note is trying to understand the conjecture in terms of modular forms and arithmetic intersection. In Section 2, we interpret the Faltings’ height from the moduli point of view as in Faltings’ original definition and relate it with Siegel modular forms and Hilbert modular forms, and arithmetic intersections. For example, we have (Corollary 2.4)

**Proposition 1.1.** Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with prime discriminant $D \equiv 1 \mod 4$. Let $K$ a quartic CM number field with real quadratic subfield $F$, and let $\Phi$ be a CM type of $K$. Let $X$ be the moduli stack of abelian surfaces with real multiplication by $\mathcal{O}_F$, and let $\mathcal{M}_k$ be the line bundle on $X$ of Hilbert modular forms of weight $k$ with the Petersson metric. Then

\[
\frac{k\# \text{CM}(K, \Phi)}{W_K} h_{\text{Fal}}(A) = h_{\text{Fal}}(\mathcal{M}(K, \Phi)).
\]

Here $W_K$ is the number of roots of unity in $K$, $\text{CM}(K, \Phi)$ is the 0-cycle of CM abelian surfaces of CM type $(\mathcal{O}_K, \Phi)$ in $\mathcal{X}(\bar{\mathbb{Q}})$, and $\mathcal{M}(K, \Phi)$ is the flat closure of $\text{CM}(K, \Phi)$ in $\mathcal{X}$. Let $\Psi$ be a normalized meromorphic Hilbert modular form for $\text{SL}_2(\mathcal{O}_F)$ of weight $k$ such that $\text{div} \, \Psi$ and $\mathcal{CM}(K, \Phi)$ intersect properly, then

\[
\frac{k\# \text{CM}(K, \Phi)}{W_K} h_{\text{Fal}}(A) = \text{div} \, \Psi \cdot \mathcal{CM}(K, \Phi) - \frac{1}{W_K} \sum_{z \in \text{CM}(K, \Phi)} ||\Psi(z)||_{\text{Pet}}
\]

for an Abelian surface of the CM type $(K, \Phi)$. 
In Section 3, we review Colmez’s conjecture and unravel his definition of class function $A_0^\Phi$ associated to a CM type $\Phi$, and prove the following proposition.

**Proposition 1.2.** Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of prime discriminant $D \equiv 1 \pmod{4}$. Let $K$ be a non-biquadratic CM quartic field with real quadratic field $F$ with a CM type $\Phi$. Then Colmez’s conjecture for the CM type $\Phi$ is the same as

$$h_{\text{Fal}}(A) = \frac{1}{2} \beta(K/F).$$

Here

$$\beta(K/F) = \frac{\Lambda'(0, \chi_{K/F})}{\Lambda(0, \chi_{K/F})} + \Gamma'(1) - \log 4\pi$$

and $\Lambda(s, \chi_{K/F})$ is the complete $L$-function of the quadratic Hecke character $\chi_{K/F}$ associated to $K/F$ as defined in (3.19). In particular, the Faltings height is independent of the choice of CM types of $K$.

Finally, let $X$ be the moduli stack over $\mathbb{Z}$ of abelian varieties $(A, \iota, \lambda)$ with real multiplications (see Section 2 for precise definition). Let $CM(K)$ be the moduli stack of $(A, \iota, \lambda)$ where $\iota: O_K \subset \text{End}(A)$ is an $O_K$-action on $A$ such that $(A, \iota|_{O_F}, \lambda) \in X$, and the Rosati involution associated to the polarizations $\lambda$ gives the complex conjugation on $K$. The map $(A, \iota, \lambda) \mapsto (A, \iota|_{O_F}, \lambda)$ is a finite proper map from $CM(K)$ into $M$, and we denote its direct image in $M$ still by $CM(K)$ by abuse of notation. Finally, let $T_m$ be the flat closure of the well-known Hirzebruch-Zagier divisors $T_m$ in $X$, see [BBK] for more information. Then $T_m$ and $CM(K)$ are arithmetic 2 and 1-cycles in the arithmetic 3-fold $X$ and they intersect properly. In [BY, (1.10)] (a minor mistake in the conjectured formula), it is conjectured

$$T_m, CM(K) = \frac{1}{2} b_m.$$

Here

$$b_m = \sum_p b_m(p) \log p$$

is defined as follows. Let $\tilde{K}$ be the reflex field of $(K, \Phi)$ with real quadratic field $\tilde{F} = \mathbb{Q}(\sqrt{\tilde{D}})$. Then

$$b_m(p) \log p = \sum_{p|p} \sum_{t = 2 \gcd(d_{\tilde{K}/\mathbb{F}_p}, |n| \sqrt{D})} B_t(p)$$

where

$$B_t(p) = \begin{cases} 0 & \text{if } p \text{ is split in } \tilde{K}, \\ (\text{ord}_p t_n + 1)\rho(td_{\tilde{K}/\mathbb{F}}p^{-1}) \log |p| & \text{if } p \text{ is not split in } \tilde{K}, \end{cases}$$

and

$$\rho(a) = \#\{A \subset O_{\tilde{K}} : N_{\tilde{K}/\mathbb{F}}A = a\}.$$

The main result of this paper is
Theorem 1.3. Let the notation be as above, and assume that \( d_K = D^2 \tilde{D} \) with \( \tilde{D} \equiv 1 \mod 4 \) being prime. Then

\[
g(\tau) = \frac{\# \text{CM}(K)}{2} (-h_{\text{Fal}}(A)) + \frac{1}{2} \beta(K/F) + \sum_{m > 0} (T_m \text{CM}(K) - \frac{1}{2} b_m) q^m
\]

is a modular form of weight 2, level \( D \), and character \( \epsilon_D = (\overline{D}) \). Moreover, Colmez’s conjecture holds for \( K \) if and only if \( g(\tau) \) is a cusp form.

We will prove this theorem in Section 4. Here is the rough idea. Bruinier, Burgos-Gil, and Künn defined an arithmetic Hirzebruch-Zagier divisors \( \hat{T}_m \) in \( X \) and proved that

\[
\hat{\phi}(\tau) = \phi(\tau) = M_1^2 + \sum_{M \geq 1} \hat{T}_m e(m \tau)
\]

is a modular form of weight 2, level \( D \), Nybentypus character \( (\overline{D}) \) with values in \( \hat{\text{CH}}^1(X) \). Doing height pairing with \( \text{CM}(K) \) gives rise to the following modular form (see (4.6))

\[
\phi(\tau) = -\frac{\# \text{CM}(K)}{W_K} h_{\text{Fal}}(A) + \sum_{m > 0} (T_m \text{CM}(K) + \frac{2}{W_K} G_m(\text{CM}(K))) q^m.
\]

On the other hand, [BY, Theorems 5.1, 8.1] (see also Theorem 4.1) asserts that

\[
f(\tau) = -\frac{\# \text{CM}(K)}{2W_K} \beta(K/F) + \sum_{m > 0} \left( \frac{1}{2} b_m + \frac{2}{W_K} G_m(\text{CM}(K)) \right) q^m
\]

is a modular form of weight 2, level \( D \), Nybentypus character \( (\overline{D}) \). Since \( g(\tau) = \phi(\tau) - f(\tau) \), one obtains the theorem.

In [Ya1], we prove that \( T_1 \text{CM}(K) = \frac{1}{2} b_1 \) if furthermore \( \mathcal{O}_K \) is a free \( \mathcal{O}_F \)-module. In particular, for \( D = 5, 13, 17 \), this, together with Theorem 1.3, implies that \( g(\tau) \) is cuspidal, and so Colmez’s conjecture holds in these cases. In [Ya2], we prove \( T_m \text{CM}(K) = \frac{1}{2} b_m \) for all \( m \geq 1 \) assuming further that \( \mathcal{O}_K \) is a free \( \mathcal{O}_F \)-module. It gives the first non-Abelian Chowla-Selberg formula.

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2. Faltings’ height

Let \( g \geq 1 \) be an integer, and let \( \mathbb{H}_g \) be the Siegel upper plane of genus \( g \), i.e., the set of symmetric matrices \( z = x + iy \in \text{Sym}_g(\mathbb{C}) \) such that \( y > 0 \) is totally positive. Let \( A_g = \text{Sp}_g(\mathbb{Z}) \backslash \mathbb{H}_g \) be the open Siegel modular variety of genus \( g \) over \( \mathbb{C} \). Let \( A_g \) be the moduli stack over \( \mathbb{Z} \) of principally polarized abelian varieties \( (A, \lambda) \), then \( A_g(\mathbb{C}) = [A_g] \) as
orbifolds. Let \( \hat{A}_g \) be a Toroidal compactification, and let \( \omega \) be the Hodge bundle on \( \hat{A}_g \). It has a natural metric defined as follows. Let \( \alpha \) be a section of \( \omega \) and let \( z = (A_z, \lambda_z) \in A_g(\mathbb{C}) \),

The value \( \alpha_z \) od \( \alpha \) at \( z \) has metric

\[
||\alpha_z||_{\text{nat}}^2 = \left| \frac{1}{2\pi i} \int_{A_z(\mathbb{Z})} \alpha \wedge \bar{\alpha} \right|
\]

We write \( \hat{\omega} = (\omega, ||||_{\text{nat}}) \) for this ‘naturally’ metrized Hodge bundle. We remark that differently authors use different normalizing factor (we use \((\frac{1}{2\pi i})^g \) here). For a primitive arithmetic 1-cycle \( \mathcal{Z} = (A, \lambda) \in A_g(\mathcal{O}_L) \) where \( L \) is a number field and \( \mathcal{O}_L \) is the ring of integers of \( L \), we define its Faltings’ height with respect to \( \hat{\omega} \) as

\[
h_{\hat{\omega}}(\mathcal{Z}) = \frac{1}{[L : \mathbb{Q}]} \text{div } s. \mathcal{Z} - \sum_{\sigma : L \hookrightarrow \mathbb{C}} \frac{1}{\# \text{Aut}(\sigma(A), \lambda)} \log ||\sigma(A), \lambda||_{\text{nat}}.
\]

Here \( \text{Aut}(\sigma(A), \lambda) \) is the automorphism group of \((\sigma(A), \lambda)\) over \( \mathbb{C} \). It does not change when we replace \( L \) by its finite extensions. We define the Faltings’ height of an arithmetic 1-cycle \( \mathcal{Z} \) by linearity. Let \( \omega_{A/L} = \wedge^g \Omega_{A/L} \) which is an invertible \( \mathcal{O}_L \)-module (since \( A \) has good reduction everywhere). Let \( L' \) be the Hilbert class field of \( L \), then \( \omega_{A/L'} = \omega_{A/L} \otimes \mathcal{O}_{L'} \) is a principal \( \mathcal{O}_{L'} \)-module. Without loss of generality, we may thus assume that \( \omega_{A/L} = \mathcal{O}_L \alpha \) is already principal. In this case (2.2) gives

\[
h_{\hat{\omega}}((A, \lambda)) = -\frac{1}{2[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \frac{1}{\# \text{Aut}(\sigma(A), \lambda)} \log \left( \frac{1}{2\pi i} \int_{\sigma(A)(\mathbb{C})} \sigma(\alpha) \wedge \overline{\sigma(\alpha)} \right).
\]

Let \((A, \iota, \lambda)\) be a CM abelian variety over \( \mathbb{C} \) of CM type \((\mathcal{O}_K, \Phi)\), i.e.,

\[
\iota : \mathcal{O}_K \hookrightarrow \text{End}(A)
\]

such that the induced action of \( \mathcal{O}_K \) on \( \Omega_A \) is given by the CM type \( \Phi \). Then \((A, \iota, \lambda)\) descends to a abelian variety \((A_L, \iota, \lambda)\) where \( A_L \) is an abelian variety over \( \mathcal{O}_L \) with good reduction everywhere, and \( \iota \) and \( \lambda \) are also defined over \( \mathcal{O}_L \). In such a case,

\[
\text{Aut}((\sigma(A_L), \lambda)) = \mu_K
\]

is the group of unity in \( K \), and is independent of the choice of \( L \) or \( \sigma : L \hookrightarrow \mathbb{C} \). So it is natural to define

\[
h_{\text{Fal}}(A) = W_K h_{\hat{\omega}}((A_L, \lambda)) = -\frac{1}{2[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \log \left( \frac{1}{2\pi i} \int_{\sigma(A_L)(\mathbb{C})} \sigma(\alpha) \wedge \overline{\sigma(\alpha)} \right).
\]

Here \( W_K = \# \mu_K \). Notice that this normalization differs from Colmez’s normalization by \( \frac{g}{2} \log 2\pi \) [Co]. It is not independent of \( L \). In fact, Colmez proved in [Co, Theorem 0.3] that it is only dependent of \((K, \Phi)\).

By [CF, Page 141], if \( f(\tau) \) is a Siegel modular form for \( \text{Sp}_g(\mathbb{Z}) \) of weight \( k \), then

\[
\alpha(f) = f(\tau)(2\pi i dw_1 \wedge 2\pi i dw_2 \wedge \cdots \wedge 2\pi i dw_g)^k
\]

is a section of \( \omega^k \), when pulling back to \( \mathbb{H}_g \), where \( dw_1 \wedge dw_2 \wedge \cdots \wedge dw_g \) is a trivialization of \( \omega_c \) over \( \mathbb{H}_g \). Moreover, \( \alpha(f) \) gives a section of \( \omega^k \) over a subring \( R \) if and only if the Fourier coefficients of \( f \) are defined over \( R \). Conversely, every section of \( \omega^k \) can be identified this
way. Let $\widehat{M}_k = (M_k, || \cdot ||_{\text{Pet}})$ be the line bundle of Siegel modular forms of weight $k$ with the following Petersson metric

\begin{equation}
||f(\tau)||_{\text{Pet}} = |f(\tau)|(4\pi)^g \det \Im(\tau)^{\frac{g}{2}}.
\end{equation}

Then it is easy to check that $f \mapsto \alpha(f)$ give an isomorphism between $\widehat{M}_k$ and $\widehat{\omega}_k^{\text{nat}}$. Indeed,

\begin{equation}
||\alpha(f)||_{\text{nat}}^2 = \frac{1}{W_K} \sum_{A_\tau \in CM(K, \Phi)} \log ||f(\tau)||_{\text{Pet}}^k.
\end{equation}

Let $K$ be a CM number field of degree $2g$ with a CM type $\Phi$, let $CM(K, \Phi)$ be the set of CM abelian varieties with CM type $(O_K, \Phi)$. We extend it to a arithmetic 1-cycle in $A_g$ over $O_L$ for some number field $L$, and denote it by $CM(K, \Phi)$. Then the following lemma is now obvious.

**Lemma 2.1.** Let $f$ be a normalized meromorphic Siegel modular form defined over $O_L$, i.e., its Fourier coefficients are all defined over $O_L$ and generate $O_L$. Assume that $\text{div} f$ and $CM(K, \Phi)$ intersect properly. Then

\begin{equation}
\frac{k\#CM(K, \Phi)}{W_K} h_{F_{\text{nd}}}(A) = h_{\text{Pet}}(CM(K, \Phi)) = \text{div} f CM(K, \Phi) - \frac{1}{W_K} \sum_{A_\tau \in CM(K, \Phi)} \log ||f(\tau)||_{\text{Pet}}
\end{equation}

for any CM abelian variety $A \in CM(K, \Phi)$. Here for $\tau \in \mathbb{H}_g$, $A_\tau = \mathbb{C}^g/L_\tau$ is its associated principally polarized abelian variety where $L_\tau = \tau \mathbb{Z}^g \oplus \mathbb{Z}^g$.

Next, let $F$ be a totally real number field of degree $g$, and let $\partial$ be its different. Let

\begin{equation}
\Gamma(f) = \text{SL}(O_F \oplus f) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(F) : a, d \in O_F, c \in f^{-1}, b \in f \}
\end{equation}

and let $X(f) = \Gamma(f) \backslash \mathbb{H}_g$ be a Hilbert modular variety. Let $\mathcal{X}(f)$ be the moduli stack of the triples $(A, \iota, \lambda)$ defined over some number field where $A$ is Abelian variety of dimension $g$ with real multiplication

\begin{equation}
\iota : O_F \subset \text{End}(A)
\end{equation}

and

\begin{equation}
\lambda : f^{-1}\partial^{-1} \rightarrow \text{Hom}_{O_F}(A, A^\vee)^{\text{Sym}}
\end{equation}

is a polarization module map satisfying the Deligne-Pappa condition (see [Go]):

\begin{equation}
f^{-1}\partial^{-1} \otimes A \rightarrow A^\vee, \quad (r, a) \mapsto \lambda(r)a
\end{equation}

is an isomorphism. Then $X(f)$ is the coarse moduli scheme of $\mathcal{X}_C$. The map $(A, \iota, \lambda) \mapsto (A, \lambda(1))$ gives a natural map from $\mathcal{X}(\partial^{-1})$ to $A_g$ which extends to a map $\phi$ from a Toroidal compactification $\bar{\mathcal{X}}(\partial^{-1})$ to some $\bar{A}_g$. Over $X(\partial^{-1} = \Gamma(\partial^{-1}) \backslash \mathbb{H}_g$, the map is given as
follows. Let \( e = \{e_1, \ldots, e_g\} \) be an ordered \( \mathbb{Z} \)-basis of \( \mathcal{O}_F \) and let \( f = \{f_1, \ldots, f_g\} \) be a basis of \( \partial^{-1} \) such that
\[
\text{tr}_{F/Q} e_i f_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]
Let \( \sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_g\} \) be the (ordered) set of real embeddings of \( F \), and set
\[
R = \sigma(e) = (\sigma_i(e_j)) \in M_g(\mathbb{R}).
\]
Then it is easy to check
\[
{}^t R^{-1} = \sigma(f) = (\sigma_i(f_j)).
\]
Finally for \( a \in \mathcal{O}_F \) and \( z = (z_1,\ldots,z_g) \in \mathbb{C}^g \), we set
\[
a^* = \text{diag}(\sigma_1(a),\ldots,\sigma_g(a)), \quad z^* = \text{diag}(z_1,\ldots,z_g).
\]

**Lemma 2.2.** Let the notation be as above, then the map
\[
\phi : \Gamma(\partial^{-1}) \backslash \mathbb{H}^g \to \text{Sp}_g(\mathbb{Z}) \backslash \mathbb{H}_g
\]
is given by
\[
\phi(z) = {}^t R z^* R.
\]
The associated map \( \Gamma(\partial^{-1}) \to \text{Sp}_g(\mathbb{Z}) \) is given by
\[
\phi\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \text{diag}(R^{-1}, {}^t R) \left( \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \right) \text{diag}(R, {}^t R^{-1}).
\]

**Proof.** Let \( \Lambda = \mathcal{O}_F \oplus \partial^{-1} \) be with the symplectic form
\[
\langle {}^t (x_1, x_2), {}^t (y_1, y_2) \rangle = \text{tr}_{F/Q}(x_1 y_2 - x_2 y_1).
\]
We embed \( F \) into \( \mathbb{R}^g \) via \( \sigma \) and then embed \( \Lambda \) into \( \mathbb{R}^{2g} = \mathbb{R}^g \oplus \mathbb{R}^g \). Then
\[
\Lambda = \text{diag}(R, {}^t R^{-1}) L,
\]
with \( L = \mathbb{Z}^g \oplus \mathbb{Z}^g \) being the standard lattice of \( \mathbb{R}^{2g} \) with the standard symplectic form. \( \Gamma(\partial^{-1}) \) acts on \( \Lambda \) linearly and preserves the symplectic form, so it acts on \( L \) and preserves its symplectic form, this gives the map \( \phi : \Gamma(\partial^{-1}) \to \text{Sp}_g(\mathbb{Z}) \) in the lemma. Indeed, for \( \gamma = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \Gamma(\partial^{-1}) \), one has
\[
\gamma L = \text{diag}(R^{-1}, {}^t R) \gamma \Lambda = \text{diag}(R^{-1}, R) \left( \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \right) \text{diag}(R, {}^t R^{-1}) L.
\]
For \( z \in \mathbb{H}^g \), its associated Abelian variety is \( A_z = \mathbb{C}^g / \Lambda_z \), where
\[
\Lambda_z = \left\{ \left( \sigma_1(a) z_1 + \sigma_1(b), \ldots, \sigma_g(a) z_g + \sigma_g(b) \right) : a \in \mathcal{O}_F, b \in \partial^{-1} \right\} = z^* \mathbb{Z}^g + {}^t R^{-1} \mathbb{Z}^g = {}^t R^{-1} L \tau.
\]
Here \( \tau = {}^t R z^* R \in \mathbb{H}_g \), and
\[
L \tau = \tau \mathbb{Z}^g + \mathbb{Z}^g = \{ \tau a + b : a, b \in \mathbb{Z}^g \}.
\]
So \( A_z \) is isomorphic to \( A_\tau \) where \( A_\tau \) is the Abelian variety associated \( \tau \in \mathbb{H}_g \). This proves the lemma. \( \square \)
Notice \(|\det R| = \sqrt{d_F}\) where \(d_F\) is the absolute discriminant of \(F\). So for a Siegel modular form \(f\) of weight \(k\),
\[
\|f(\phi(z))\|_{\text{Pet}}^2 = |f(\phi(z))|^2(4\pi)^{\frac{3}{2}} Rz^* R) = |f(\phi(z))|^2(4\pi)^{\frac{3}{2}}d_F \prod \text{Im}(z_i)^k.
\]

Let \(\mathcal{M}_k(\vartheta^{-1})\) be the line bundle of Hilbert modular forms of weight \(k\) on \(X(\vartheta^{-1})\), and let \(\mathcal{M}_k(\vartheta^{-1}) = (\mathcal{M}_k(\vartheta^{-1}), \|\|_{\text{Pet}}\) be the metrized line bundle of Hilbert modular forms of weight \(k\) with the following Petersson metric
\[
(2.9) \quad \|\Psi(z)\|_{\text{Pet}}^2 = |\Psi(z)|^2(4\pi)^{\frac{3}{2}}d_F \prod \text{Im}(z_i)^k.
\]
It can be extended to a metrized line bundle on \(\tilde{X}(\vartheta^{-1})\), which we still donate by \(\mathcal{M}_k(\vartheta^{-1})\).
Notice that for a CM number field \(K\) with maximal totally subfield \(F\), \(\mathcal{C}M(K, \Phi)\) can be viewed as an arithmetic 1-cycle in \(\mathcal{X}(\vartheta^{-1})\). So we have

**Corollary 2.3.** Let \(\Psi\) be a normalized meromorphic Hilbert modular form for \(\Gamma(\vartheta^{-1})\) of weight \(k\) such that \(\text{div} \, \Psi\) intersect with \(\mathcal{C}M(K, \Phi)\) properly. Then
\[
k\# \mathcal{C}M(K, \Phi) W_K h_{\text{Fal}}(A) = h_{\mathcal{M}_k(\vartheta^{-1})}(\mathcal{C}M(K, \Phi)) = \text{div} \, \Psi \mathcal{C}M(K, \Phi) - \frac{1}{W_K} \sum_{z \in \mathcal{C}M(K, \Phi)} \log \|\Psi(z)\|_{\text{Pet}}.
\]

Now we consider a special case which is in the main interest of this paper. Let \(F = \mathbb{Q}(\sqrt{D})\) be a real quadratic field with discriminant \(D \equiv 1 \pmod{4}\) being a prime number. In this case,
\[
\Gamma := \Gamma(O_F) \cong \Gamma(\vartheta^{-1}), \quad \gamma \mapsto \tilde{\gamma} = \text{diag}(1, \frac{\sqrt{D}}{\epsilon}) \gamma \text{diag}(1, \frac{\epsilon}{\sqrt{D}}),
\]
where \(\epsilon > 1\) is a fundamental unit of \(F\) so that \(\epsilon \epsilon' = -1\). This induces an isomorphism
\[
\Gamma \backslash H^2 \rightarrow \Gamma(\vartheta^{-1}) \backslash H^2, (z_1, z_2) \mapsto (\frac{\epsilon}{\sqrt{D}} z_1, \frac{-\epsilon'}{\sqrt{D}} z_2).
\]

Let \(\hat{\mathcal{M}}_k = (\mathcal{M}_k, \|\|_{\text{Pet}})\) be the metrized line bundle on \(\tilde{X}\) of Hilbert modular forms for \(\Gamma = \text{SL}_2(O_F)\) with the following Petersson metric
\[
(2.10) \quad \|\Psi(z)\|_{\text{Pet}} = |\Psi(z)|(16\pi^2 y_1 y_2)^{\frac{3}{2}}
\]
Then the above remark and Corollary 2.3 gives

**Corollary 2.4.** Let \(\Psi\) be a normalized meromorphic Hilbert modular form for \(\text{SL}_2(O_F)\) of weight \(k\) such that \(\text{div} \, \Psi\) and \(\mathcal{C}M(K, \Phi)\) intersect properly, then
\[
k\# \mathcal{C}M(K, \Phi) W_K h_{\text{Fal}}(A) = h_{\hat{\mathcal{M}}_k}(\mathcal{C}M(K, \Phi)) = \text{div} \, \Psi \mathcal{C}M(K, \Phi) - \frac{1}{W_K} \sum_{z \in \mathcal{C}M(K, \Phi)} \|\Psi(z)\|_{\text{Pet}}
\]
for an Abelian surface of the CM type \((K, \Phi)\).
3. The Colmez Conjecture

In this section, we review Colmez’s conjecture [Co] and pay special attention in the end for the case $K$ is a quartic CM number field.

We fix an embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$, and view all number fields as subfields of $\mathbb{Q}$. Let $\mathbb{Q}^{CM}$ be the composite of all CM number fields in $\mathbb{Q}$. It has a unique complex conjugation $\rho$. For a CM number field $L$, we denote $G_{L}^{CM} = \text{Gal}(\mathbb{Q}^{CM}/L)$ and simply $G_{CM}^{\mathbb{Q}}$. We define the Haar measure on $G_{CM}^{L}$ with total volume 1, i.e.,

$$\int_{G_{CM}^{L}} d\mu = \text{Vol}(G_{CM}^{L}) = 1.$$ 

So $\text{Vol}(G_{L}^{CM}) = [L : \mathbb{Q}]^{-1}$.

For a field $R$ of characteristic 0, let $H(G_{CM}^{L}, R)$ be the Hecke algebra of $G_{CM}^{L}$, i.e., the ring (without identity) of locally constant functions $\Phi$ on $G_{CM}^{L}$ with values in $R$ with the convolution as the multiplication:

$$\Phi_{1} \ast \Phi_{2}(g) = \int_{G_{CM}^{L}} \Phi_{1}(h)\Phi_{2}(h^{-1}g)dh.$$ 

When $R = \mathbb{C}$ (or a subfield invariant under the complex conjugation), we define the reflex function $\Phi^{\vee}$ via $\Phi^{\vee}(g) = \Phi(\rho g^{-1})$, and define a positive definite Hermitian form

$$\langle \Phi_{1}, \Phi_{2} \rangle = \int_{G_{CM}^{L}} \Phi_{1}(h)\overline{\Phi_{2}(h)}dh = (\Phi_{1} \ast \Phi_{2}^{\vee})(1).$$

Let $H^{0}(G_{CM}^{L}, R)$ be the subring of locally constant class functions on $G_{CM}^{L}$ with values in $R$, i.e., $\Phi \in H(G_{CM}^{L}, R)$ such that

$$\Phi(hgh^{-1}) = \Phi(g) \quad \text{for all } g, h \in G_{CM}^{L}.$$ 

By Brauer’s theorem, $H^{0}(G_{CM}^{L}, \mathbb{Q})$ has a basis given by all Artin characters $\chi = \chi_{\pi}$ of $G_{CM}^{L}$, where $\pi$ runs over all irreducible representations of $G_{CM}^{L}$. For an Artin character $\chi$ of $G_{CM}^{L}$, we denote by $f_{\text{Art}}(\chi)$ the analytic Artin conductor (i.e., the one used for the functional equation), $L(s, \chi)$ the Artin L-function, and define

$$Z(s, \chi) = \frac{L'(s, \chi)}{L(s, \chi)}, \quad \mu_{\text{Art}}(\chi) = \log f_{\text{Art}}(\chi).$$

We extend the definition linearly to all functions $\Phi \in H^{0}(G_{CM}^{L}, \mathbb{Q})$.

Notice that there is a projection map $\Phi \mapsto \Phi^{0}$ from $H(G_{CM}^{L}, \mathbb{Q})$ to $H^{0}(G_{CM}^{L}, \mathbb{Q})$, given by

$$\Phi^{0}(g) = \int_{G_{CM}^{L}} \Phi(hgh^{-1})dh = \sum_{\chi} \langle \Phi, \chi \rangle \chi.$$

A CM type is a function $\Phi \in H(G_{CM}^{L}, \mathbb{Z})$ such that $\Phi(g) \in \{0, 1\}$ and

$$\Phi(g) + \Phi(\rho g) = 1 \quad \text{for every } g \in G_{CM}^{L}.$$
This is consistent with the usual definition of a CM type. Indeed, let $K$ be a subfield of finite degree over $\mathbb{Q}$ such that
\begin{equation}
\Phi(gh) = \Phi(g) \quad \text{for all } h \in G_K^{CM}, g \in G^{CM},
\end{equation}
then $\Phi$ can be viewed as a formal sum
\begin{equation}
\Phi = \sum_{\sigma: K \hookrightarrow \overline{\mathbb{Q}}} a_{\sigma}(\Phi)\sigma,
\end{equation}
where $a_{\sigma}(\Phi) = \Phi(g)$ for any $g \in G^{CM}$ with $g|K = \sigma$. The two conditions on a CM type function $\Phi$ is exactly what is needed to make the formal sum $\Phi$ a CM type of $K$ in the usual sense. Conversely, a formal sum as (3.8) gives rise to a function $\Phi$ on $G^{CM}$. We will use the same notation $\Phi$ for the two meanings of a CM type. If we take $K$ to be the smallest subfield of $\mathbb{Q}^{CM}$ such that (3.7) holds, then $(K, \Phi)$ is a primitive CM type. When $K$ is Galois over $\mathbb{Q}$, the reflex type $\tilde{\Phi}$ in the usual sense corresponds to the reflex function $\Phi^\vee$.

For a CM type $\Phi$, we define
\begin{equation}
A_\Phi = \Phi \ast \Phi^\vee
\end{equation}
and let $A_\Phi^0$ be the projection of $A_\Phi$ to $H^0(G^{CM}, \overline{\mathbb{Q}})$. Concretely, let $(K, \Phi)$ be a CM type of a CM number field $K$ in the usual sense, and let $M$ be a CM Galois extension of $\mathbb{Q}$ containing $K$, and let $\Phi_M = \sum_{\sigma|K} a_{\sigma} \sigma$ be the extension of $\Phi$. Then
\begin{equation}
A_\Phi = \frac{1}{[M: \mathbb{Q}]} \Phi_M \Phi_M
\end{equation}
Here we recall that $\Phi_M = \sum a_{\sigma} \sigma^{-1}$ if $\Phi_M = \sum a_{\sigma} \sigma$. Moreover, if
\begin{equation}
A_\Phi = \sum_{\sigma \in \text{Gal}(M/\mathbb{Q})} c(\sigma)\sigma
\end{equation}
then
\begin{equation}
A_{\Phi}^0 = \sum_{\sigma} c^0(\sigma)\sigma, \quad \text{with } c^0(\sigma) = \frac{1}{\#[\sigma]} \sum_{\tau \in [\sigma]} c(\tau).
\end{equation}
Here $[\sigma]$ is the conjugacy class of $\sigma$ in $\text{Gal}(M/\mathbb{Q})$.

Let $(K, \Phi)$ be a CM type, and let $A$ be a CM abelian variety of CM type $(O_K, \Phi)$, we may assume that $A$ is defined over a number field $L$ with good reduction everywhere. Let $h_{\text{Fal}}(A)$ be the Faltings’ height of $A$. It can be proved that
\begin{equation}
h_{\text{Fal}}(\Phi) = \frac{1}{[K: \mathbb{Q}]} h_{\text{Fal}}(A)
\end{equation}
is independent of the choices of $A$ and is even independent of the choice of $K$ if we view $\Phi$ as a function of $G^{CM}$. We call it the Faltings’s height of $\Phi$. [Co, Theorem 0.3] asserts that
there is a unique $\mathbb{Q}$-linear function $\text{ht}$—the height function—from $H^0(G^CM, \mathbb{R})$ satisfying a specific condition and

\begin{equation}
(3.14) \quad h_{\text{Fal}}(\Phi) = -\text{ht}(A^0_\Phi) - \frac{1}{2} \mu_{\text{Art}}(A^0_\Phi) + \frac{1}{4} \log 2\pi.
\end{equation}

Here the extra term $\frac{1}{4} \log 2\pi$ is due to the different normalization of the Faltings' height between our definition and Colmez's. Furthermore, he conjectured [Co, Conjecture 0.4] that For any $\Phi \in H^0(G^CM, \mathbb{Q})$, one

\begin{equation}
(3.15) \quad \text{ht}(\Phi) = Z(0, \Phi^\vee).
\end{equation}

In terms of Faltings's height, it means

**Conjecture 3.1.** (Colmez) $h_{\text{Fal}}(\Phi) = -Z(0, A^0_\Phi) - \frac{1}{2} \mu_{\text{Art}}(A^0_\Phi) + \frac{1}{4} \log 2\pi$.

Two CM types $\Phi_1$ and $\Phi_2$ are called equivalent if there is $\tau \in G^CM$ such that

$\Phi_1(\sigma) = \Phi_2(\tau \sigma)$

for every $\sigma \in G^CM$. Clearly, two equivalent CM types have the same Faltings' height.

Now we consider some simple examples. First let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with the CM type $\Phi = \sigma_0$ where $\sigma_0$ is the identity map. In this case, $\tilde{\Phi} = \Phi$ and

$A^0_\Phi = A^0_\Phi = \frac{1}{2} \sigma_0 = \frac{1}{4} (\chi_0 + \chi_{-d})$

where $\chi_0$ is the trivial character and $\chi_{-d}$ is the Dirichlet quadratic character associated $K/\mathbb{Q}$. So the Colmez conjecture is simply

\begin{equation}
(3.16) \quad 2h_{\text{Fal}}(E) = -\frac{\zeta'(0)}{\zeta(0)} - \frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log d + \log 2\pi = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log d
\end{equation}

for a CM elliptic curve with CM by $\mathcal{O}_{-d}$. This is a reformulation of the Chowla-Selberg formula (1.1) [Co].

Next let $K = \mathbb{Q}(\sqrt{D}, \sqrt{-d})$ be a bi-quadratic CM number field with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ and two imaginary quadratic field $F_1 = \mathbb{Q}(\sqrt{-d})$ and $F_2 = \mathbb{Q}(\sqrt{-Dd})$. Let $\Phi = 1 + \sigma$ be a CM type of $K$ with 1 being the identity map and $\sigma(\sqrt{D}) = -\sqrt{D}$, $\sigma(\sqrt{-d}) = -\sqrt{-d}$, i.e., $\sigma$ fixes $F_1$. Then $\tilde{\Phi} = \Phi$, and

$A^0_\Phi = A_\Phi = \frac{1}{4} (1 + \sigma)^4 = \frac{1}{2} (1 + \sigma) = \frac{1}{4} (\chi_0 + \chi_{-d})$

where $\chi_0$ is the trivial character of $\text{Gal}(K/\mathbb{Q})$ and $\chi_{-d}$ is the nontrivial character of $\text{Gal}(F_1/\mathbb{Q})$ viewed as a character of $\text{Gal}(K/\mathbb{Q})$. So the Colmez conjecture inserts

\begin{equation}
(3.17) \quad h_{\text{Fal}}(A) = -\frac{\zeta'(0)}{\zeta(0)} - \frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log d + \log 2\pi = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log d
\end{equation}

for a CM abelian surface of CM type $(\mathcal{O}_K, \Phi)$ by. That is the same as $h_{\text{Fal}}(A) = 2h_{\text{Fal}}(E)$ for a CM elliptic curve with CM by $\mathcal{O}_{-d} = \mathbb{Z}[\sqrt{-d + \sqrt{-d}}]$, by the Chowla-Selberg formula (3.16). Indeed, let $E$ be a CM elliptic curve with CM by $\mathcal{O}_{-d}$. Then $A = E \otimes \mathcal{O}_D \cong E \times E$.
is of CM type \((K, \Phi)\) with CM by \(\mathcal{O}_K\). So \(h_{\text{Fal}}(A) = 2h_{\text{Fal}}(E)\). We summarize the two examples as the following proposition in comparison with the next proposition.

**Proposition 3.2.** Let \(K = \mathbb{Q}(\sqrt{D}, \sqrt{-d})\) be a bi-quadratic CM number field. Let \(E\) be a CM elliptic curve with CM by \(\mathcal{O}_d\) and let \(A\) be a CM abelian surface of CM type \((\mathcal{O}_K, \Phi)\), where \(\Phi = \text{Gal}(K/\mathbb{Q}(\sqrt{-d}))\). Then

\[
h_{\text{Fal}}(A) = 2h_{\text{Fal}}(E) = \frac{1}{2} \Gamma'(1) - \frac{1}{2} \log 4\pi + \frac{\Lambda'(0, \chi_{-d})}{\Lambda(0, \chi_{-d})}.
\]

Here

\[
\Lambda(s, \chi_{-d}) = (\frac{d}{\pi})^{s+1/2} \Gamma\left(\frac{s}{2} + 1\right)L(s, \chi_{-d}).
\]

Now let \(K = F(\sqrt{\Delta})\) be a non-biquadratic quartic CM number field with maximal totally real subfield \(F = \mathbb{Q}(\sqrt{\Delta})\), the case of special interest in this paper. We first assume that \(K\) is not Galois over \(\mathbb{Q}\), and let \(M\) be the smallest Galois extension of \(\mathbb{Q}\) containing \(K\). Then \(\text{Gal}(M/\mathbb{Q}) = <\sigma, \tau>\) is a dihedral group \(D_4\) with

\[
\sigma(\sqrt{\Delta}) = \sqrt{\Delta'}, \quad \sigma(\sqrt{\Delta'}) = -\sqrt{\Delta},
\]

\[
\tau(\sqrt{\Delta}) = \sqrt{\Delta'}, \quad \tau(\sqrt{\Delta'}) = \sqrt{\Delta}.
\]

Let \(\Phi = 1 + \sigma\) be a CM type of \(K\), and let \(\tilde{K}\) be its reflex field with maximal real quadratic field \(\tilde{F}\). Then

\[
\Phi_M = 1 + \sigma + \tau \sigma + \tau, \quad \tilde{\Phi}_M = 1 + \sigma^{-1} + \tau \sigma + \tau.
\]

So

\[
A_\Phi = \frac{1}{8} \Phi_M \tilde{\Phi}_M = \frac{1}{4} (2 + 2\tau + \sigma + \sigma^{-1} + \tau \sigma + \sigma \tau),
\]

and

\[
A_\Phi^0 = \frac{1}{4} (1 - \rho + \sum_{\alpha \in \text{Gal}(M/\mathbb{Q})} \alpha) = \frac{1}{4} (\frac{1}{2} \chi_\pi + \chi_1).
\]

Here \(\chi_0\) is the trivial character of \(G_\mathbb{Q}\), and \(\pi\) is the unique 2-dimensional representation of \(\text{Gal}(M/\mathbb{Q})\), and can be realized as \(\text{Ind}_{G_\mathbb{Q}}^{G_{\tilde{F}}} \chi_{K/\tilde{F}}\), where \(\chi_{K/\tilde{F}}\) is the nontrivial quadratic character of \(G_{\tilde{F}}\) factoring through \(\text{Gal}(\tilde{K}/\tilde{F})\). Notice that \(\pi\) is also \(\text{Ind}_{G_\mathbb{Q}}^{G_F} \chi_{K/F}\).

When \(K\) is cyclic quartic CM field with a CM type \(\Phi\). The same calculation (slightly simpler) shows

\[
A_\Phi^0 = \frac{1}{4} (\frac{1}{2} \chi_\pi + \chi_0)
\]

as above with \(\pi = \text{Ind}_{G_\mathbb{Q}}^{G_F} \chi_{K/F}\). Notice that \(\pi\) is not irreducible in this case.

Let \(\chi = \chi_{K/\tilde{F}}\) be the quadratic Hecke character of \(\tilde{F}\) associated to \(K/\tilde{F}\), and let

\[
\Lambda(s, \chi) = (f_\chi)^2 \pi^{-s-1} \Gamma\left(\frac{s + 1}{2}\right)^2 L(s, \chi)
\]

be the complete L-function of \(\chi\) as defined in [BY], Section 6, where \(f_\chi = N_{F/Q} d_{K/\tilde{F}} d_{\tilde{F}}\) is the Artin conductor of \(\chi\). Then we have
Proposition 3.3. Let $K$ be a non-biquadratic quartic CM number field with real quadratic subfield $F$. Let $\Phi$ be a CM type of $K$ and let $\tilde{K}$ be its reflex field with real quadratic field $\tilde{F}$. Let $\chi = \chi_{\tilde{K}/\tilde{F}}$ be as above. Then the Colmez conjecture for $\Phi$ is the same as

\begin{equation}
8h_{\text{Fal}}(\Phi) = -\frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} + \Gamma'(1) - \log 4\pi.
\end{equation}

That is,

\begin{equation}
\label{eq:3.21}
h_{\text{Fal}}(A) = \frac{1}{2} \beta(\tilde{K}/\tilde{F}) = \frac{1}{2} \beta(K/F)
\end{equation}

for any CM abelian surface with CM by $\mathcal{O}_K$.

Proof. The above calculation gives

$$A_0^{\Phi} = \frac{1}{8} \chi_{\pi} + \frac{1}{4} \chi_0$$

where $\chi_{\pi}$ is the character of the two dimensional representation $\pi = \text{Ind}_{G_{\tilde{F}}}^{G_F} \chi_{\tilde{K}/\tilde{F}}$, and $\chi_0$ is the trivial character. So

$$Z(s, \chi_{\pi}) = \frac{L'(s, \chi_{\tilde{K}/\tilde{F}})}{L(s, \chi_{\tilde{K}/\tilde{F}})}, \quad \mu_{\text{Art}}(\chi_{\pi}) = \log f_{\chi_{\tilde{K}/\tilde{F}}},$$

we trust the reader will distinguish the representation $\pi$ from the number $\pi$). For $\chi = \chi_{\tilde{K}/\tilde{F}}$, one has

$$8 \left( Z(0, A_0^{\Phi}) + \frac{1}{2} \mu_{\text{Art}}(A_0^{\Phi}) \right) = \frac{L'(0, \chi)}{L(0, \chi)} + 2 \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2} \log f_\chi = \frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} + \log \pi - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + 2 \frac{\zeta'(0)}{\zeta(0)}.$$

Now recall that

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = -\gamma - 2 \log 2, \quad \frac{\zeta'(0)}{\zeta(0)} = \log 2\pi, \quad \Gamma'(1) = -\gamma,$$

where $\gamma$ is the Euler constant. So

$$8 \left( Z(0, A_0^{\Phi}) + \frac{1}{2} \mu_{\text{Art}}(A_0^{\Phi}) \right) = \frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} + \gamma + \log 4\pi + 2 \log(2\pi).$$

Now the proposition is clear. Notice that $\beta(\tilde{K}/\tilde{F}) = \beta(K/F)$, and $L(s, \chi_{\tilde{K}/\tilde{F}}) = L(s, \chi_{K/F})$ since $\text{Ind}_{G_{\tilde{F}}}^{G_F} \chi_{\tilde{K}/\tilde{F}} = \text{Ind}_{G_{\tilde{F}}}^{G_F} \chi_{K/F} = \pi$ as explained above.

\[ \square \]

This proposition implies that $h_{\text{Fal}}(A)$ for a CM abelian surface with CM by $\mathcal{O}_K$ is independent of choice of the CM abelian surface or the CM type when $K$ is non-biquadratic. This is different from the bi-quadratic case discussed above. It might be interesting to note

$$h_{\text{Fal}}(A_d) + h_{\text{Fal}}(A_{Dd}) = -\frac{\Lambda'(0, \chi_{K/F})}{\Lambda(0, \chi_{K/F})} + \Gamma'(1) - \log 4\pi$$
for the bi-quadratic case $K = \mathbb{Q}(\sqrt{D}, \sqrt{-d})$, which is very much like Proposition 3.3. Here $A_d$ (resp. $A_{dd}$) is a CM abelian surface of the CM type $\Phi_d = \text{Gal}(K/\mathbb{Q}(\sqrt{-d}))$ (resp. $\Phi_{dd} = \text{Gal}(K/\mathbb{Q}(\sqrt{-Dd}))$, and $F = \mathbb{Q}(\sqrt{D})$.

4. Proof of the main theorem

The purpose of this section is to prove Theorem 1.3. First we recall a modularity result of Bruinier, Burgos-Gil and Kühn on arithmetic Hirzebruch-Zagier divisors.

Let $\tilde{X}$ be a Toroidal compactification of the arithmetic Hilbert modular surface $X$ and let $\tilde{T}_m$ be the corresponding compactification of $T_m$ in $\tilde{X}$. Bruinier, Burgos-Gil and Kühn defined in [BBK] an Arakelov divisor $\hat{T}_m = (\tilde{T}_m, 2G_m) \in \hat{CH}^1(\tilde{X})$ (we prefer a slightly different renormalization so that $(\Psi, -\log \|\|)^{2}$ is a principal divisor for a rational function on $\tilde{X}$). They proved in the same paper [BBK, Theorem A] that

$$\hat{\phi}^{\tau} = M_{\frac{1}{2}} + \sum_{m \geq 1} \hat{T}_m e(m\tau) \in M_{\frac{1}{2}}(D, (\frac{D}{-})) \otimes \hat{CH}^1(\tilde{X})$$

is a modular form valued in $\hat{CH}^1(\tilde{X})$ for $\Gamma_0(D)$ of weight 2 with Nebentypus character $(\frac{D}{-})$.

Here $M_{\frac{1}{2}}(D, (\frac{D}{-}))$ is the subspace of modular forms of weight 2, level $D$, and Nebentypus character $(\frac{D}{-})$ such that its Fourier expansion $f(\tau) = \sum_{m \geq 0} a_m e(m\tau)$ satisfies $a_m = 0$ if $(\frac{D}{m}) = -1$.

Recall that there is a bilinear form—the Faltings’ height pairing

$$\hat{CH}^1(\tilde{X}) \times \mathbb{Z}^2(\tilde{X}) \to \mathbb{C}$$

which is given by

$$h(\tau, G)(Z) = \tau . Z + \frac{1}{2} \sum_{z \in \mathbb{Z}^2(C)} \frac{1}{\# \text{Aut}(z)} G(z)$$

when $\mathcal{T}$ and $\mathcal{Z}$ intersect properly.

Let $\mathcal{CM}(K)$ be the moduli stack over $\mathbb{Z}$ representing the moduli problem which assigns a base scheme $S$ to the set of the triples $(A, \iota, \lambda)$ where $\iota : O_K \hookrightarrow \text{End}_S(A)$ is a CM action of $O_K$ on $A$, and $(A, \iota|_{O_F}, \lambda) \in \mathcal{M}(S)$ such that the Rosati involution associated to $\lambda$ induces to the complex conjugation of $O_K$. The map $(A, \iota, \lambda) \mapsto (A, \iota|_{O_F}, \lambda)$ is a finite proper map from $\mathcal{CM}(K)$ into $\mathcal{M}$, and we denote its direct image in $\mathcal{M}$ still by $\mathcal{CM}(K)$ by abuse of notation. It was proved in [Ya1, Lemma 2.1] that

$$\mathcal{CM}(K)(\mathbb{C}) = 2\text{CM}(K) := 2(\text{CM}(K, \Phi) + \text{CM}(K, \Phi'))$$

where $\Phi = \{1, \sigma\}$ and $\Phi' = \{1, \sigma^{-1}\}$ are CM types of $K$ given in Section 3. As mentioned in Section 2, $h_{\text{Fal}}(A)$ depends only on its CM type. Since $A \mapsto \sigma^{-1}(A)$ is a bijection between $\text{CM}(K, \Phi)$ and $\text{CM}(K, \Phi')$, and $h_{\text{Fal}}(A) = h_{\text{Fal}}(\sigma^{-1}(A))$, we have

$$h_{\text{Fal}}(\mathcal{CM}(K)) = 2\# \text{CM}(K) h_{\text{Fal}}(A)$$
for a CM Abelian surface $A$ with CM by $\mathcal{O}_K$. By Corollary 2.4, one sees that
\begin{equation}
(4.5) \quad h_{\mathcal{M}^+}(\mathcal{CM}(K)) = -\frac{1}{W_K} \#\text{CM}(K) h_{\text{Fal}}(A).
\end{equation}

Now applying the height paring function to $\hat{\phi}(\tau)$ and $\mathcal{CM}(K)$, one obtain the following modular form in $M_2(D, (\frac{D}{-}))$
\begin{equation}
(4.6) \quad \phi(\tau) = -\frac{\#\text{CM}(K)}{W_K} h_{\text{Fal}}(A) + \sum_{m>0} (\frac{1}{2} b_m + \frac{2}{W_K} G_m(\text{CM}(K))) q^m.
\end{equation}
Here
\[ G_m(\text{CM}(K)) = \sum_{z \in \text{CM}(K)} G_m(z). \]

This is the first main step in proving Theorem 1.3. To continue, we need a result of Bruinier and Yang [BY] on modularity of CM values of automorphic Green functions $G_m$, which we state as

**Theorem 4.1. (Bruinier and Yang)** The function
\[ f(\tau) = -\frac{\#\text{CM}(K)}{2W_K} \beta(K/F) + \sum_{m>0} \left( \frac{1}{2} b_m + \frac{2}{W_K} G_m(\text{CM}(K)) \right) q^m \]
is a modular form belonging to $M_2^+(D, (\frac{D}{-}))$. Here $b_m$ are the constants defined in the introduction.

**Proof.** (basic idea) Let
\[ E^+_2(\tau) = 1 + \sum_{m>0} C(m, 0) q^m \in M_2^+(D, (\frac{D}{-})) \]
be the (unique) normalized Eisenstein series in $M_2(D, (\frac{D}{-}))$ defined in [BY, Corollary 2.3]. Using derivative of incoherent Hilbert Eisenstein series, diagonal restriction (to elliptic modular forms), and holomorphic projection, Bruinier and the author proved ([BY, Theorem 8.1]) that
\begin{equation}
(4.7) \quad F(\tau) = \sum_{m>0} \left( \frac{1}{2} b_m + \frac{1}{2} c_m \right) q^m + \frac{1}{4} \Lambda(0, \chi_{\tilde{K}/\tilde{F}}) \beta(\tilde{K}/\tilde{F}) (E^+_2(\tau, 0) - 1)
\end{equation}
is a cusp form belonging to $S_2^+(D, (\frac{D}{-}))$, where
\[ c_m = \lim_{s \to 1} \{ 2 \sum_{a=0}^{\frac{n+m\sqrt{D}}{2}} \rho(td_{\tilde{K}/\tilde{F}})Q_{s-1}(\frac{n}{m\sqrt{D}}) 
\quad \times \Lambda(0, \chi_{\tilde{K}/\tilde{F}}) \left( \frac{C(m, 0)}{2(s-1) - L_m} \right). \]

Here $d_{\tilde{K}/\tilde{F}}$ is the relative discriminant of $\tilde{K}/\tilde{F}$, the subscript + means the totally positive, $\rho(a) = \#\{ \mathfrak{A} \subset \mathcal{O}_{\tilde{K}} : N_{\tilde{K}/\tilde{F}}(\mathfrak{A}) = a \}$.
is the norm counting function, $\mathcal{L}_m$ is some normalizing constant depending on $m$, and $Q_{s-1}(t)$ is the so-called the Legendre function of the second kind.

On the other hand, the CM value of $G_m$ is given by [BY, Theorem 5.1] (together with normalization in [BY, (2.24), (2.25)]) that

$$
\frac{2}{W_K} G_m(CM(K, \Phi))
= \lim_{s \to 1} \left[ \sum_{\mu = \frac{n-m\sqrt{q}}{2p} \in \mathcal{L}_m} Q_{s-1} \left( \frac{n}{m\sqrt{q}} \right) \rho_{\tilde{K}/\tilde{F}}(\mu d_{\tilde{K}/\tilde{F}}) \right.
+ \left. \frac{C(m,0)}{2(s-1)} - \mathcal{L}_m \right] \frac{\#CM(K, \Phi)}{W_K}.
$$

One has the same formula for $G_m(CM(K', \Phi'))$ with $\frac{2-m\sqrt{D}}{2D}$ replaced by $\frac{n+m\sqrt{D}}{2D}$. So

$$
\frac{4}{W_K} G_m(CM(K)) = c_m + \lim_{s \to 1} \left( \frac{C(m,0)}{2(s-1)} - \mathcal{L}_m \right) \left( \frac{2\#CM(K)}{W_K} - \Lambda(0, \chi_{\tilde{K}/\tilde{F}}) \right).
$$

This implies

$$
\frac{4}{W_K} G_m(CM(K)) = c_m, \quad \Lambda(0, \chi_{\tilde{K}/\tilde{F}}) = \frac{2\#CM(K)}{W_K}.
$$

Plugging this into (4.7) and using the facts (see proof of Proposition 3.3)

$$
\Lambda(s, \chi_{\tilde{K}/\tilde{F}}) = \Lambda(s, \chi_{K/F}), \quad \beta(\tilde{K}/\tilde{F}) = \beta(K/F)
$$

and

$$
W_K = W_{\tilde{K}} = \begin{cases} 
10 & \text{if } K = \mathbb{Q}(\zeta_5), \\
2 & \text{otherwise},
\end{cases}
$$

one obtains the theorem. \(\square\)

**Proof of Theorem 1.3:** Now the proof of Theorem 1.3 is clear. Indeed, one has by (4.6) and Theorem 4.1

$$
g(\tau) = \phi(\tau) - f(\tau).
$$

**References**


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