

# ON CM ABELIAN VARIETIES OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. In this paper, we associate canonically to every imaginary quadratic field  $K = \mathbb{Q}(\sqrt{-D})$  one or two isogenous classes of CM (complex multiplication) abelian varieties over  $K$ , depending on whether  $D$  is odd or even ( $D \neq 4$ ). These abelian varieties are characterized as of smallest dimension and smallest conductor, and such that the abelian varieties themselves descend to  $\mathbb{Q}$ . When  $D$  is odd or divisible by 8, they are the scalar restriction of ‘canonical’ elliptic curves first studied by Gross and Rohrlich. We prove that these abelian varieties have the striking property that the vanishing order of their  $L$ -function at the center is dictated by the root number of the associated Hecke character. We also prove that the smallest dimension of a CM abelian variety over  $K$  is exactly the ideal class number of  $K$  and classify when a CM abelian variety over  $K$  has the smallest dimension.

**0. Introduction.** Let  $K = \mathbb{Q}(\sqrt{-D})$  be an imaginary quadratic field with fundamental discriminant  $-D$  and ideal class number  $h$ . According to Rohrlich ([Ro2]), a Hecke character  $\chi$  of  $K$  of conductor  $\mathfrak{f}$  is called *canonical* if it satisfies the following three conditions:

- (0.1)  $\chi(\bar{\mathfrak{a}}) = \overline{\chi(\mathfrak{a})}$  for all ideals  $\mathfrak{a}$  of  $K$  prime to  $\mathfrak{f}$ ;
- (0.2)  $\chi(\alpha\mathcal{O}_K) = \pm\alpha$  for every principal ideal  $\alpha\mathcal{O}_K$  of  $K$  prime to  $\mathfrak{f}$ ;
- (0.3) The conductor  $\mathfrak{f}$  is divisible only by primes ramified in  $K/\mathbb{Q}$ .

The canonical Hecke characters have the following amazing properties.

(A) The associated  $L$ -function  $L(s, \chi)$  behaves extremely well at the symmetric center. Indeed, one has ([MR] for the value and [MY] for the derivative)

$$(0.4) \quad \text{ord}_{s=1} L(s, \chi) = \frac{1 - W(\chi)}{2}.$$

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Here  $W(\chi)$  is the root number of  $\chi$ —the sign of the functional equation relating  $L(s, \chi)$  to  $L(2 - s, \chi)$ .

Clearly, if  $\chi$  is a canonical Hecke character of  $K$ , so is  $\chi\phi$  for every ideal class character  $\phi$  of  $K$ . Moreover,  $\chi_H = \chi \circ N_{H/K}$  depends only on the family  $\{\chi\phi\}$ , and gives rise to a CM elliptic curve  $E(\chi)$  over  $H$ , unique up to  $H$ -isogeny. Here and in the rest of this paper,  $H$  is the Hilbert class field of  $K$ . We call such a CM elliptic curve over  $H$  canonical.

(B) Rohrlich proved in [Ro2] that canonical Hecke characters exist if and only if  $D$  is odd or  $8|D$ . When  $D$  is odd, there is a unique  $H$ -isogeny class of canonical CM elliptic curves over  $H$ . When  $8|D$ , there are exactly two  $H$ -isogeny classes of canonical CM elliptic curves over  $H$ . So in some sense, Rohrlich associated canonically one or two  $H$ -isogeny classes of CM elliptic curves over  $H$  to every  $K = \mathbb{Q}(\sqrt{-D})$  with  $2 \nmid D$  or  $8|D$ . In fact, when  $D$  is an odd prime, Gross ([Gr1-2]) pinned down further a special elliptic curve  $A(D)$  over  $F = \mathbb{Q}(j(\frac{1+\sqrt{-D}}{2}))$  and studied its arithmetic extensively. In general, combining (0.4) with the well-known works of Gross-Zagier ([GZ]), Kolyvagin and Logachev ([KL]), one sees that canonical CM elliptic curves have finite Shafarevich-Tate group over  $H$  and have the Mordell-Weil rank (over  $H$ ) 0 or  $2h$  depending solely on whether  $W(\chi) = +1$  or  $-1$ . Notice that the scalar restriction of a canonical CM elliptic curve from  $H$  to  $K$  is a CM abelian variety over  $K$  of dimension  $h$ , and descends to an abelian variety over  $\mathbb{Q}$ .

A main purpose of this paper is to extend the above result to the case  $4||D$ . Instead of CM elliptic curves, we have to consider CM abelian varieties over  $K$ . We call a CM abelian variety  $(A, i)$  over  $K$  (of CM type  $(T, \Phi)$ ) *simplest* if

(0.6) The abelian variety  $A$  itself descends to an abelian variety over  $\mathbb{Q}$ .

(0.7)  $A$  has the smallest possible dimension and analytic conductor subject to the condition (0.6). By the analytic conductor of  $A$ , we mean the constant  $N$  appeared in the functional equation for the  $L$ -function  $L(s, A)$ .

A Hecke character of  $K$  is *simplest* if it is the Hecke character of  $K$  associated to a simplest CM abelian variety over  $K$ . Equivalently, a Hecke character of  $K$  is simplest if and only if it satisfies (0.1), and

$$(0.2') \quad \chi(\alpha\mathcal{O}_K) = \epsilon(\alpha)\alpha \text{ for a character } \epsilon : (\mathcal{O}_K/\mathfrak{f})^* \longrightarrow \mathbb{C}^*.$$

(0.3')  $\chi$  has the smallest possible conductor subject to (0.1) and (0.2'), and the field  $\mathbb{Q}(\chi)$  generated by all  $\chi(\mathfrak{a})$  has the smallest possible degree over  $\mathbb{Q}$ , again subject to (0.1) and (0.2').

In section 4, we will prove

**Theorem 0.1.** *Let  $K = \mathbb{Q}(\sqrt{-D})$  be a quadratic imaginary quadratic field with*

fundamental discriminant  $-D$ , and let

$$n(D) = \begin{cases} 1 & \text{if } D \text{ is odd or } D = 4, \\ 2 & \text{if } D > 4 \text{ is even.} \end{cases}$$

Then there are exactly  $n(D)$  simplest CM abelian varieties  $A$  over  $K$  of dimension  $h$ , up to  $K$ -isogeny. Moreover, when  $D$  is odd or  $8 \parallel D$ , they are the scalar restriction from  $H$  to  $K$  of the canonical CM elliptic curves over  $H$ .

As in the case of canonical Hecke characters, we actually constructed the simplest Hecke characters in Section 4. The dimension of a simplest CM abelian variety over  $K$  is always  $h$ , and the conductor of a simplest Hecke character  $\chi$  is

$$\begin{cases} \sqrt{-D}\mathcal{O}_K & \text{if } 2 \nmid D, \\ \sqrt{-D}\mathfrak{p}_2 & \text{if } 4 \parallel D, \\ 2\sqrt{-D}\mathcal{O}_K & \text{if } 8 \mid D. \end{cases}$$

Here  $\mathfrak{p}_2$  is the prime ideal of  $K$  above 2.

There are some new features in the case  $4 \parallel D$  worth mentioning. Firstly, the simplest CM abelian varieties over  $K$  are *not* scalar restriction of any CM elliptic curves over  $H$  when  $4 \parallel D$ . Secondly, If  $(A, i)$  is a simplest CM abelian variety over  $K$  of CM type  $(T, \Phi)$ , and  $\chi$  is the associated simplest Hecke character of  $K$ , then for any  $\sigma \in \Phi$ ,  $\chi^\sigma$  is also a simplest Hecke character of  $K$  associated to  $(A, i \circ \sigma^{-1})$  of CM type  $(T^\sigma, \Phi \circ \sigma^{-1})$ . The abelian variety  $A$  is really associated to the family of simplest Hecke characters  $\{\chi^\sigma : \sigma \in \Phi\}$ . while  $D$  is odd or  $8 \mid D$ , this family coincide with the family  $\{\chi\phi : \phi \text{ is an ideal class character of } K\}$ , it is not the case when  $4 \parallel D$ . This leads some complication in computing the central  $L$ -value and derivative. Anyway, In sections 5, we compute the root numbers of *simplest* Hecke characters and prove in sections 6 and 7 that the amazing formula (0.4) holds for all simplest Hecke characters of  $K$ , which we record here as

**Theorem 0.2.** *Let  $\chi$  be a simplest Hecke character of  $K = \mathbb{Q}(\sqrt{-D})$ . Then (0.4) holds.*

So simplest CM abelian varieties behave just as well as the canonical CM elliptic curves. In particular, they have finite Shafarevich-Tate group over  $K$  and their Mordell-Weil rank over  $K$  is either 0 or  $2h$  depending on whether the root number is  $+1$  or  $-1$ .

In view of the definition of simplest CM abelian varieties, it is natural to ask what is the smallest dimension of a CM abelian variety over  $K$ . We will prove in Section 3

**Theorem 0.3.** *Let  $(A, i)$  be a CM abelian variety over  $K$  of CM type  $(T, \Phi)$ , and let  $\chi$  be the associated Hecke character of  $K$  satisfying (0.2'). Then*

- (1) (Theorem 3.1) *one has  $h \mid \dim A$ .*
- (2) (Theorem 3.5) *The equality  $\dim A = h$  holds if and only if one of the following holds.*
  - (a)  *$\epsilon$  is quadratic,*
  - (b)  *$3 \mid D$  and  $\epsilon$  is of order 6, there is  $\alpha \in K^*$  such that  $3\alpha^2$  is prime to  $\mathfrak{f}$  and  $\epsilon(3\alpha^2) = -1$ , or*
  - (c)  *$4 \mid D$  and  $\epsilon$  is of order 4 or 12, there is  $\alpha \in K^*$  such that  $2\alpha^2$  is prime to  $\mathfrak{f}$  and  $\epsilon(2\alpha^2) = \pm i$ .*

*In particular, whether  $A$  has the smallest dimension  $h$  depends solely on the behavior of  $\chi$  on the principal ideals, i.e., the behavior of  $\epsilon$ . When  $\dim A = h$ ,  $A$  is a scalar restriction of a CM elliptic curve over the Hilbert class field  $H$  of  $K$  if and only if  $\text{Im}(\epsilon) \subset \mathcal{O}_K^*$ .*

For simplicity, we sometimes leave out the CM type of CM abelian varieties in this paper. In Section 1, We review the basic relation between CM abelian varieties and the algebraic Hecke characters. In Section 2, we study the relation between the scalar restriction of CM elliptic curves over  $H$  and  $h$ -dimensional CM abelian varieties over  $K$ . In particular, we prove (Proposition 2.2) that a CM abelian variety over  $K$  of dimension  $h$  is a scalar restriction of a CM elliptic curve over  $H$  if and only if its Hecke character has values in  $K$  on principal ideals.

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## 1. CM abelian varieties and algebraic Hecke characters.

Let  $K$  be a number field. Then a type of  $K$  is a formal sum  $\Phi = \sum_{\sigma: K \hookrightarrow \mathbb{C}} n_\sigma \sigma$  with  $n_\sigma \in \mathbb{Z}$ . If  $L$  is a finite extension of  $K$ , one can extend  $\Phi$  to a type  $\Phi_L$  of  $L$  via

$$\Phi_L = \sum_{\sigma: L \hookrightarrow \mathbb{C}} n_\sigma \sigma, \quad n_\sigma = n_{\sigma|_K}.$$

A type  $\Phi$  is called simple if it can not be extended from a proper subfield. When  $K$  is Galois over  $\mathbb{Q}$ , embeddings of  $K$  into  $\mathbb{C}$  are elements of the Galois group

$\text{Gal}(K/\mathbb{Q})$ . In this case, one defines its reflex type

$$\Phi' = \sum_{\sigma: K \hookrightarrow \mathbb{C}} n_{\sigma} \sigma^{-1}.$$

In general, given a type  $(K, \Phi)$ , its reflex type  $(K', \Phi')$  is defined as follows. Firstly,  $K'$  is the subfield of  $\mathbb{C}$  generated by all  $\alpha^{\Phi} = \prod \sigma(\alpha)^{n_{\sigma}}$ ,  $\alpha \in K^*$ . Secondly, let  $L$  be a finite Galois extension of  $\mathbb{Q}$  containing both  $K$  and  $K'$ , extend  $\Phi$  to  $\Phi_L$ . It is then a standard fact that there is a unique type  $\Phi'$  of  $K'$  such that its extension to  $L$  is

$$(\Phi')_L = (\Phi_L)'.$$

This  $\Phi'$  is independent of the choice of  $L$  and is called the reflex type of  $\Phi$ .

An algebraic Hecke character of  $K$  of infinite type  $\Phi$  and modulus  $\mathfrak{f}$  (an integral ideal) is a group homomorphism

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^*$$

such that for every  $\alpha \equiv 1 \pmod{\mathfrak{f}}$

$$\chi(\alpha \mathcal{O}_K) = \alpha^{\Phi}.$$

Here  $I(\mathfrak{f})$  denotes the group of fractional ideals of  $K$  prime to  $\mathfrak{f}$ , and  $\alpha \equiv 1 \pmod{\mathfrak{f}}$  means

$$\text{ord}_v(\alpha - 1) \geq \text{ord}_v \mathfrak{f} \quad \text{if } \text{ord}_v \mathfrak{f} > 0,$$

and  $\sigma(\alpha) > 0$  for every real embedding  $\sigma$  of  $K$ . The Dirichlet unit theorem implies that there is an algebraic Hecke character of  $K$  of infinite type  $\Phi$  if and only if  $w(\Phi) = n_{\sigma} + n_{\rho\sigma}$  is independent of the choice of  $\sigma$ , where  $\rho$  is the complex conjugation of  $\mathbb{C}$ . In such a case,  $\Phi$  is called a Serre type of weight  $w(\Phi)$ . Notice that the subfield  $\mathbb{Q}(\chi)$  of  $\mathbb{C}$  generated by  $\chi(\mathfrak{a})$ ,  $\mathfrak{a} \in I(\mathfrak{f})$ , is a number field containing the reflex field  $K'$  of  $(K, \Phi)$ . We say  $\chi$  has values in  $T$  if  $\mathbb{Q}(\chi) \subset T$ .

When  $K$  is a CM number field, i.e., a quadratic totally imaginary extension of a totally real number field, a type  $\Phi$  of  $K$  is a CM type if  $n_{\sigma} \geq 0$  and  $n_{\sigma} + n_{\rho\sigma} = 1$  for every complex embedding  $\sigma$  of  $K$ . In this case,  $\Phi$  is often identified with the set of embeddings  $\{\sigma : n_{\sigma} = 1\}$ . In general, any extension of a CM type just defined is also called a CM type.

An abelian variety  $A$  defined over a subfield  $L$  of  $\mathbb{C}$  is said to be a CM abelian variety over  $L$  if there is a number field  $K$  of degree  $[K : \mathbb{Q}] = 2 \dim A = 2d$  together with an embedding

$$i : K \hookrightarrow \text{End}_L^0 A = \text{End}_L A \otimes \mathbb{Q}.$$

In such a case,  $K$  acts on the differentials  $\Omega_{A/\mathbb{C}}$  diagonally via  $d$  embeddings  $\Phi = \{\phi_1, \dots, \phi_d\}$ : there is a basis  $\omega_i$  for  $\Omega_{A/\mathbb{C}}$  such that for every  $\alpha \in i^{-1}(\text{End}_L A)$

$$i(\alpha)^*(\omega_i) = \phi_i(\alpha)\omega_i.$$

We will identify  $\Phi$  with the formal sum  $\sum_{\phi \in \Phi} \phi$ . We usually call  $(A, i)$  is of CM type  $(K, \Phi)$ . It is a fact ([Sh2, Theorem 1, Page 40]) that the two seemingly different definitions of CM types are the same. We remark that the CM type of  $(A, i)$  depends on  $i$ ,  $A$  can have different CM types if one allows  $i$  change. For example, let  $A = E^2$  be the square of a CM elliptic curve  $E$  by a quadratic field  $K$ . Then  $\text{End}^0 A = M_2(K)$ , and any quadratic field extension of  $K$ , embedded into  $M_2(K)$ , give rise to CM type of  $A$ .

The following theorem summarizes the basic relation between CM abelian varieties and algebraic Hecke characters.

**Theorem 1.1.** *Let  $K \subset \mathbb{C}$  be a number field, and let  $(T, \Phi)$  be a CM type and  $(T', \Phi')$  be its reflex type. Assume  $T' \subset K$ .*

(1) *(Shimura-Taniyama) If  $(A, i)$  is a CM abelian variety over  $K$  of CM type  $(T, \Phi)$ , then there is a (unique) algebraic Hecke character  $\chi$  of  $K$  of infinite type  $\Phi'_K$  such that  $i(\chi(\mathfrak{p}))$  reduces to the Frobenius endomorphism of  $A$  modulo  $\mathfrak{p}$  for every prime ideal  $\mathfrak{p}$  where  $A$  has good reduction. In particular,*

$$L(s, A/K) = \prod_{\sigma: T \hookrightarrow \mathbb{C}} L(s, \chi^\sigma).$$

Here  $\chi^\sigma = \sigma \circ \chi$ . We call  $\chi$  the associated (algebraic) Hecke character of  $(A, i)$  or simply  $A$ .

(2) *(Casselman) Conversely, if  $\chi$  is an algebraic Hecke character of  $K$  of infinite type  $\Phi'_K$ , valued in  $T$ , then there is a CM abelian variety  $(A, i)$  over  $K$  of CM type  $(T, \Phi)$ , unique up to isogeny, such that the associated Hecke character of  $(A, i)$  is  $\chi$ . We call  $(A, i)$  or simply  $A$  an associated abelian variety of  $\chi$ .*

(3) *Let  $(A, i)$  be a CM abelian variety over  $K$  of CM type  $(T, \Phi)$  and let  $\chi$  be the associated Hecke character of  $K$ . Then the following are equivalent.*

(a) *The CM abelian variety  $A$  is simple over  $K$ .*

(b)  *$T = \mathbb{Q}(\chi)$ .*

*Proof.* For (1), see for example [Sh2, Theorems 19.8 and 19.11]. For (2), see for example [Sh2, Theorem 21.4]. If  $A$  is not simple, then  $A$  is isogenous to  $B^r$  for some simple abelian variety  $B$  over  $K$ , and  $B$  is a CM abelian variety of some CM type  $(T_1, \Phi_1)$ . In this case,  $\chi_A = j \circ \chi_B$ , where  $j$  is an embedding of  $T_1$  to  $T$ . So  $\mathbb{Q}(\chi_A) \subset T_1 \neq T$ . Conversely, if  $T_1 = \mathbb{Q}(\chi_A) \neq T$ , let  $(B, i_B)$  be a CM abelian variety over  $K$  of CM type  $(T_1, \Phi_1)$  associated to  $\chi_A$ , where  $\Phi_1 = \Phi|_{T_1}$ .

Then  $(B^r, i'_B)$  is of CM type  $(T, \Phi)$  for some  $i'_B$  where  $r = [T : T_1]$ . So  $(A, i_A)$  and  $(B^r, i'_B)$  have the same CM type and the same Hecke character, and thus are  $K$ -isogenous to each other.  $A$  is thus not simple over  $K$ . This proves (3)

*Remark 1.2.* If  $(A, i)$  is a CM abelian variety over  $K$  of CM type  $(T, \Phi)$ , with algebraic Hecke character  $\chi$ , then  $(A, i \circ \sigma^{-1})$  is of type  $(T^\sigma, \Phi \circ \sigma^{-1})$ , with algebraic Hecke character  $\chi^\sigma$ . When  $A$  is simple over  $K$ ,  $(T^\sigma, \Phi \circ \sigma^{-1})$  and  $i \circ \sigma^{-1}$  are all possible CM types and embeddings. In particular, the arithmetic and L-function of  $A$  over  $K$  do not depend on particular choices of the CM types or the embeddings. the abelian variety  $A$  is in correspondence with the family of algebraic Hecke characters  $\{\chi^\sigma : \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})\}$ . If one wants to leave the infinite type  $\Phi'_K$  of  $\chi$  unchanged, one needs to require  $\sigma \in \text{Aut}(\mathbb{C}/K'')$  where  $K''$  is the reflex field of  $(K, \Phi'_K)$ . However, if  $A$  descends to a subfield  $F$  of  $K$  not containing the reflex field of  $(T, \Phi)$ , the the  $L$ -function of  $A$  over  $F$  does depend on the choice of  $(T, \Phi)$ .

## 2. CM abelian varieties over a quadratic field and Scalar restriction of CM elliptic curves.

From now on, Let  $K = \mathbb{Q}(\sqrt{-D})$  be an imaginary quadratic field with fundamental discriminant  $-D$ , let  $H$  be the Hilbert class field of  $K$ , and let  $h = [H : K]$  be the ideal class number of  $K$ . Let  $E$  be a CM elliptic curve over  $H$  of CM type  $(K, id)$ , where  $id$  is the prefixed complex embedding of  $K$  such that  $\text{Im}(\sqrt{-D}) > 0$ . Let  $\chi_E$  be the algebraic Hecke character of  $H$  associated to  $E$ , with values in  $K$ . It is interesting to study the scalar restriction  $B = \text{Res}_{H/K} E$ , which is an abelian variety over  $K$  such that  $B(R) = E(H \otimes_K R)$  for every  $K$ -algebra  $R$ . In order for  $B$  to be a CM abelian variety over  $K$ ,  $E$  has to be a  $K$ -curve, in the sense that every Galois conjugate  $E^\sigma$  for  $\sigma \in \text{Gal}(H/K)$  is  $H$ -isogenous to  $E$ . But this condition is not sufficient in general. We refer to [Gr1] and [Na] for results and references in this direction. In particular, the following proposition is a special case of [Na, Theorems 1 and 2].

**Proposition 2.1.** *Let the notation be as above. Then  $B = \text{Res}_{H/K} E$  is a CM abelian variety over  $K$  if and only if there is an algebraic Hecke character  $\chi$  of  $K$  such that  $\chi_E = \chi \circ N_{H/K}$ .*

According to [Sh2, Theorem 19.13], the condition  $\chi_E = \chi \circ N_{H/K}$  is also equivalent to that  $K(E_{\text{tor}})$  is the maximal abelian extension of  $K$ .

**Proposition 2.2.** *Let  $(A, i)$  be a simple CM abelian variety over  $K$  of CM type  $(T, \Phi)$ . Let  $\chi_A$  be the associated algebraic Hecke character of  $K$  with values in  $T$ . Then the following are equivalent.*

- (1) *There is a CM elliptic curve  $E$  over  $H$  such that  $A$  is isogenous to the scalar restriction of  $E$  from  $H$  to  $K$ .*

(2) One has  $\chi_A(\alpha\mathcal{O}_K) \in K^*$  for every principal ideal of  $K$  prime to the conductor of  $\chi_A$ .

*Proof.* (1)  $\Rightarrow$  (2). The assumption (1) asserts that  $\chi_H = \chi_A \circ N_{H/K}$ . By the global class field theory, every ideal  $\alpha\mathcal{O}_K$  is equal to  $\beta N_{H/K}\mathfrak{A}$  for some element  $\beta$  of  $K$  congruent to 1 mod  $\mathfrak{f}$  and some ideal  $\mathfrak{A}$  of  $H$ . So  $\chi_A(\alpha\mathcal{O}_K) = \chi_E(\mathfrak{A}) \in K^*$ .

(2)  $\Rightarrow$  (1). Let  $\chi = \chi_A \circ N_{H/K}$ . By the global class field theory again,  $N_{H/K}(\mathfrak{A})$  is a principal ideal of  $K$  for every ideal  $\mathfrak{A}$  of  $H$ . So

$$\chi(\mathfrak{A}) = \chi_A(N_{H/K}(\mathfrak{A})) \in K^*,$$

and thus  $\chi$  is an algebraic character of  $H$  with values in  $K$ . Therefore there is a CM elliptic curve  $E$  over  $H$  whose associated algebraic Hecke character is  $\chi$ . One has

$$L(s, E/H) = L(s, \chi)L(s, \bar{\chi}) = \prod_{\phi} L(s, \chi_A\phi)L(s, \overline{\chi_A\phi}).$$

Here the product runs over all ideal class characters  $\phi$  of  $K$ . On the other hand, since  $A$  is simple, one has  $T = K(\chi_A)$  is generated by  $\chi_A$  over  $K$ . So one has by assumption (2)

$$\begin{aligned} \{\chi_A^\sigma : \sigma \in \Phi\} &= \{\chi_A^\sigma : \sigma \in \text{Aut}(\mathbb{C}/K)\} \\ &= \{\chi_A\phi : \phi \text{ is an ideal class character of } K\}. \end{aligned}$$

So  $L(s, A/K) = L(s, E/H) = L(s, \text{Res}_{H/K}E)$ . So  $A$  is isogenous to  $\text{Res}_{H/K}E$ , proving (1).

### 3. CM abelian varieties over a quadratic imaginary field of smallest dimension.

Let  $K = \mathbb{Q}(\sqrt{-D})$  be again a quadratic imaginary field of fundamental discriminant  $-D$ . Let  $(A, i)$  be a CM abelian variety over  $K$  of type  $(T, \Phi)$ , and let  $\chi = \chi_A$  be the associated algebraic Hecke character of  $K$  of conductor  $\mathfrak{f}$ . Then there is a character  $\epsilon$  of  $(\mathcal{O}/\mathfrak{f})^*$  such that

$$(3.1) \quad \chi(\alpha\mathcal{O}) = \epsilon(\alpha)\alpha$$

for every principal ideal of  $K$  prime to  $\mathfrak{f}$ . Obviously  $\epsilon(\alpha) = \alpha^{-1}$  for every unit  $\alpha$  in  $K$ . In particular  $\epsilon(-1) = -1$ .

Let  $\mathbb{Q}(\chi)$  be the subfield of  $\mathbb{C}$  generated by all  $\chi(\mathfrak{a})$  over  $\mathbb{Q}$ ,  $\mathfrak{a} \in I(\mathfrak{f})$ . Then  $\mathbb{Q}(\chi) \subset T$ , and  $\mathbb{Q}(\chi) = T$  if and only if  $A$  is  $K$ -simple.

**Theorem 3.1.** *Let  $(A, i)$  be a CM abelian variety over  $K$  of type  $(T, \Phi)$ . Then  $h \mid \dim A$ .*

When  $\dim A = 1$ , this gives the well-known fact that in order to have a CM elliptic curve over a quadratic imaginary field, the field has to have ideal class number 1. The simple reason is the fact that the  $j$ -invariant generates the Hilbert class field.

*Proof.* We may assume that  $A$  is simple and thus  $T = K(\chi)$ . For any number field  $F$ , let  $\mu(F)$  be the group of roots of unity in  $F$ , and let  $w_F = \#\mu(F)$ . Choose an integer  $n > 0$  such that  $w_T | n$  and  $\chi(\mathfrak{a}^n) \in K^*$  for every  $\mathfrak{a} \in I(\mathfrak{f})$ . Then

$$\theta : I(\mathfrak{f})/P(\mathfrak{f}) \longrightarrow K^*/K^{*n}, \quad [\mathfrak{a}] \mapsto \chi(\mathfrak{a}^n)K^{*n}$$

is a well-defined map and is injective. Indeed, for any principal ideal  $\alpha\mathcal{O} \in P(\mathfrak{f})$ , one has

$$\chi(\alpha\mathcal{O})^n = \epsilon(\alpha)^n \alpha^n = \alpha^n \in K^{*n},$$

which implies that  $\theta$  is well-defined. On the other hand, if  $\chi(\mathfrak{a}^n) = \alpha^n$  for some  $\alpha \in K$ . Then  $\mathfrak{a}^n$  and  $\alpha^n$  generates the same ideal in  $K$ , so are  $\mathfrak{a}$  and  $\alpha$ . Therefore  $\theta$  is injective! Clearly, the image of  $\theta$  is in the kernel of the natural map

$$K^*/K^{*n} \longrightarrow T^*/T^{*n}.$$

So one has  $h|[T : K]$  by [Ro1, Proposition 1] .

The rest of this section is to determine all CM abelian varieties over  $K$  of the smallest possible dimension  $h$ . For this purpose, we assume  $D > 4$  and fix an odd character

$$(3.2) \quad \epsilon : (\mathcal{O}_K/\mathfrak{f})^* \longrightarrow \mathbb{C}^*, \quad \epsilon(-1) = -1.$$

Let  $\chi$  be an algebraic Hecke character of  $K$  satisfying (3.1), and let  $T = K(\chi)$ , and

$$\Phi = \{\sigma : T \hookrightarrow \mathbb{C} : \sigma|_K = id\}.$$

Let  $A = A(\chi)$  be the associated simple CM abelian variety over  $K$  of type  $(T, \Phi)$ .

Let  $L = K(\epsilon)$  be the subfield of  $T$  generated by  $\epsilon$  over  $K$ . Let

$$H(\mathfrak{f}) = \{\mathfrak{a} \in I(\mathfrak{f}) : \mathfrak{a}^2 \text{ is principal } \}.$$

Then  $H(\mathfrak{f})/P(\mathfrak{f})$  is the genus ideal class group of  $K$  and has order  $2^r$ , where  $r + 1$  is the number of prime factors of  $D$ . Let  $T_g$  be the subfield of  $T$  generated by  $\chi(\mathfrak{a})$ ,  $\mathfrak{a} \in H(\mathfrak{a})$ . We have

$$(3.3) \quad K \subset L \subset T_g \subset T.$$

The same proof as in [Ro1, Theorem 2] yields

**Proposition 3.2.** (1) *One has*

$$\begin{aligned} & \{\chi^\sigma : \sigma \in \text{Gal}(\bar{\mathbb{Q}}/T_g)\} \\ &= \{\chi^\sigma : \sigma \text{ is a complex embedding of } T \text{ which is trivial on } T_g\} \\ &= \{\chi\phi : \phi \text{ is an ideal class character trivial on the genus subgroup}\}. \end{aligned}$$

(2)  $[T : K] = 2^{-r}h[T_g : K]$  depends only on the effect of  $\chi$  on ideals whose square is principal, where  $r + 1$  is the number of prime factors of  $D$ .

(3)  $\mu_T \subset T_g$ .

*Proof.* (1) The first identity is clear. For a complex embedding  $\sigma$  of  $T$  fixing  $T_g$ ,  $\phi = \chi^\sigma \chi^{-1}$  is obviously an ideal class character trivial on the genus subgroup. So it suffices to prove  $\#I(\mathfrak{f})/H(\mathfrak{f}) \leq [T : T_g]$  for the second equality. Let  $F = T_g(\mu_T)$ , and

$$\theta : I(\mathfrak{f}) \longrightarrow F^*/F^{*n}, \mathfrak{a} \mapsto \chi(\mathfrak{a}^n)F^{*n}.$$

Here  $n$  again satisfies  $\#\mu_T|n$  and  $\chi(\mathfrak{a})^n \in K^*$  for every ideal  $\mathfrak{a}$  of  $K$ . Then its image is in the kernel of the natural map  $F^*/F^{*n} \longrightarrow T^*/T^{*n}$ , and thus has order dividing  $[T : F]$  by [Ro1, Proposition 1]. Next, define

$$\psi : \ker \theta \longrightarrow K^*/K^{*n}, \mathfrak{a} \mapsto \chi(\mathfrak{a}^n)K^{*n}.$$

Then its image is in the kernel of the natural map  $K^*/K^{*n} \longrightarrow F^*/F^{*n}$ . Since  $F$  is an abelian extension of  $K$ , [Ro1, Proposition 2] implies that for any  $\mathfrak{a} \in \ker \theta$

$$\chi(\mathfrak{a}^n)^2 \in K^{*n}.$$

So  $\mathfrak{a}^2$  is principal, and thus  $\ker \theta \subset H(\mathfrak{f})$ . This proves

$$(3.4) \quad \#I(\mathfrak{f})/H(\mathfrak{f}) \leq \text{Im}(\theta) \leq [T : F] \leq [T : T_g].$$

This proves the second identity of (1), and that the inequalities in (3.4) are all equalities. In particular,

$$[T : T_g] = \#I(\mathfrak{f})/H(\mathfrak{f}) = 2^{-r}h,$$

which proves claim (2). One has also  $F = T_g$ , which gives claim (3).

**Lemma 3.3.** *Let, for any prime  $p$ ,*

$$(3.4) \quad \mu_p(\epsilon) = \{x \in \text{Im}(\epsilon) : x^{p^n} = 1 \text{ for some } n \geq 0\}.$$

*Then there is a root of unity  $c_p \in \mu_2(\epsilon)$  for each prime number  $p|D$  such that  $T_g = L(\sqrt{c_p p}, p|D)$ . Moreover, if  $\xi$  is a generator of  $\mu_2(\epsilon)$ , then  $T_g \subset L(\sqrt{\xi}, \sqrt{p}, p|D)$ .*

*Proof.* Let  $\mathfrak{p}_l$  be the prime ideal of  $K$  above  $l$ , for every  $l|D$ , then  $\mathfrak{p}_l$  has order 2 in the ideal class group of  $K$  and generate the genus class group of  $K$ . For each  $l|D$ , choose  $\alpha_p \in K^*$  so that  $\mathfrak{a}_l = \alpha_l \mathfrak{p}_l$  is relatively prime to  $\mathfrak{f}$ . Then  $\mathfrak{a}_l$  generate  $H(\mathfrak{f})/P(\mathfrak{f})$ . Now

$$\mathfrak{a}_l^2 = \alpha_l^2 l \mathcal{O} \quad \text{and} \quad \chi(\mathfrak{a}_l) = \pm \sqrt{\epsilon(\alpha_l^2 l) l} \alpha_l \in T_g.$$

So

$$(3.5) \quad \sqrt{\epsilon(\alpha_l^2 l) l} \in T_g.$$

Obviously,  $\epsilon(\alpha_l^2 l)$  can be replaced by a root of unity  $c_l \in \mu_2(\epsilon)$ . Since  $T_g$  is generated by  $\chi(\mathfrak{a}_l)$  over  $L$ , we have

$$L(\sqrt{c_l l}, l|D) = T_g.$$

Since  $\sqrt{c_l} \in L(\sqrt{\xi})$ , one has thus  $T_g \subset L(\sqrt{\xi}, \sqrt{l}, l|D)$ . This proves the lemma.

**Theorem 3.4.** *Let  $\epsilon$  be an odd character given by (3.2). Let  $\chi$  be a Hecke character of  $K$  satisfying (3.1), and let  $T = K(\chi)$ .*

(1) *If  $[T : K] = h$ , then*

$$(3.6) \quad \text{Im}(\epsilon) \subset \begin{cases} \mu_2 & \text{if } (6, D) = 1, \\ \mu_4 & \text{if } 3 \nmid D, \\ \mu_6 & \text{if } 2 \nmid D, \\ \mu_{12} & \text{if } 6|D. \end{cases}$$

(2) *Conversely, assume that  $\epsilon$  satisfies (3.6), then  $[T : K] = h$  or  $2h$ , with  $[T : K] = h$  if and only if  $\sqrt{\xi} \notin T_g$ . Here  $\xi$  is a generator of  $\mu_2(\epsilon)$ .*

(3) *Whether  $[T : K] = h$  (equivalently  $\dim A(\chi) = h$ ) or not depends only on  $\epsilon$ , i.e., the restriction of  $\chi$  on principal ideals.*

*Proof.* By Proposition 3.2,  $[T : K] = h$  if and only if  $[T_g : K] = 2^r$ . Let  $H_g = K(\sqrt{p^*}, p|D)$  be the genus class field of  $K$ , where

$$p^* = \begin{cases} (-1)^{\frac{p-1}{2}} p & \text{if } p \neq 2, \\ \frac{-D}{\prod_{p|D, p \neq 2} p^*} & \text{if } p = 2. \end{cases}$$

Notice that  $[H_g : K] = 2^r$ , where  $r + 1$  is the number of prime factors of  $D$ . Since

$$[L(\sqrt{\xi}, \sqrt{p}, p|D) : T_g] = 1 \text{ or } 2$$

by Lemma 3.3, and

$$[K(\sqrt{\xi}, \sqrt{p}, p|D) : H_g] \geq 2,$$

one has

$$\begin{aligned} [T_g : K] &\geq \frac{1}{2}[L(\sqrt{\xi}, \sqrt{p}, p|D) : K] \\ &= 2^{r-1}[L(\sqrt{\xi}, \sqrt{p}, p|D) : K(\sqrt{\xi}, \sqrt{p}, p|D)][K(\sqrt{\xi}, \sqrt{p}, p|D) : H_g] \\ &\geq 2^r[L(\sqrt{\xi}, \sqrt{p}, p|D) : K(\sqrt{\xi}, \sqrt{p}, p|D)]. \end{aligned}$$

So  $[T_g : K] \geq 2^r$ , and  $[T_g : K] = 2^r$  if and only if the following three conditions hold.

$$(3.7) \quad \sqrt{\xi} \notin T_g,$$

$$(3.8) \quad L(\sqrt{\xi}, \sqrt{p}, p|D) = K(\sqrt{\xi}, \sqrt{p}, p|D),$$

$$(3.9) \quad [K(\sqrt{\xi}, \sqrt{\pm p}, p|D) : K(\sqrt{p^*}, p|D)] = 2.$$

It is easy to see from (3.9) that  $\xi$  has at most order 4, which occurs only when  $4|D$ . (3.8) implies then that  $L(\sqrt{\xi}, \sqrt{p}, p|D) \subset K(\sqrt{-1}, \sqrt{2}, \sqrt{p}, p|D)$  contains at most  $\mu_{24}$ . So  $\mu_T = \mu_{T_g} \subset \mu_{12}$  by (3.7). A slightly more careful inspection shows (3.6).

Now assume that (3.6) is satisfied. Then  $L \subset H_g$  except for two cases:  $D = 8$  when  $\epsilon$  has order 4, and  $D = 24$  when  $\epsilon$  has order 12. So except for the two cases one has

$$[L(\sqrt{\xi}, \sqrt{p}, p|D) : H_g] = [H_g(\sqrt{\xi}) : H_g] = 2.$$

Thus  $[T_g : K] = 2^r$  or  $2^{r+1}$ . Moreover  $[T_g : K] = 2^r$  if and only if  $\sqrt{\xi} \notin T_g$ . This proves (2) except for the two exceptional cases. In the exceptional cases,  $T_p = L$  has degree  $2h$  over  $K$ , and  $\sqrt{\xi} \in T_g$ , the claim still holds. Finally, since  $c_i = \epsilon(\alpha_i^2 l)$  up to a square in  $L^*$ ,  $T_g$  is determined by  $\epsilon$ , thanks to Lemma 3.3. Now (3) follows from (2).

**Theorem 3.5.** *Let the notation be as in Theorem 3.4, and assume that  $\epsilon$  satisfies (3.6). Let  $A(\chi)$  be the CM abelian variety over  $K$  associated to  $\chi$ .*

(1) *If  $\epsilon^2 = 1$ , then  $\dim A(\chi) = h$ , and  $A(\chi)$  is isogenous to the scalar restriction of a CM elliptic curve  $E$  over  $H$ , the Hilbert class field of  $K$ .*

(2) *If  $\epsilon$  has order 4, then  $\dim A(\chi) = h$  if and only if  $4|D$ ,  $D \neq 8$ , and for some (and every) element  $\alpha_2 \in K^*$  such that  $2\alpha_2^2$  is prime to  $\mathfrak{f}$ , one has  $\epsilon(2\alpha_2^2)$  is of order 4.*

(3) *If  $\epsilon$  has order 6, then  $\dim A(\chi) = h$  if and only if  $3|D$ , and for some (and every) element  $\alpha_3 \in K^*$  such that  $3\alpha_3^2$  is prime to  $\mathfrak{f}$  and  $\epsilon(3\alpha_3^2)$  is of order 2 or 6.*

(4) If  $\epsilon$  has order 12, then  $\dim A(\chi) = h$  if and only if  $12|D$ , and for some (and every) element  $\alpha_2 \in K^*$  such that  $2\alpha_2^2$  prime to  $\mathfrak{f}$  and  $\epsilon(2\alpha_2^2)$  is of order 4 or 12.

Clearly, in cases (2)-(4) with  $D > 4$ ,  $A(\chi)$  is *not* a scalar restriction of any elliptic curve when its dimension is  $h$ .

*Proof of Theorem 3.5.* (1) follows from Proposition 2.2. For (2), notice first that  $\mu_2(\epsilon)$  is generated by  $i$  and that  $4|D$  is necessary by Theorem 3.4(1). Assume thus  $4|D$ . If  $\epsilon(2\alpha_2^2) = \pm 1$ , then  $\sqrt{\pm 2} = \sqrt{\epsilon(2\alpha_2^2)2} \in T_g$  and thus  $\zeta_8 = \frac{1}{\sqrt{2}}(1+i) \in T_g$ . This implies  $\dim A(\chi) = [T : K] = 2h$  by Theorem 3.4(2). If  $\epsilon(2\alpha_2^2) = \pm i$ , then  $\sqrt{\epsilon(2\alpha_2^2)2} = \sqrt{\pm 2i} = \pm(1 \pm i)$ , and so

$$T_g = (i, \sqrt{-D/4}, \sqrt{\epsilon_l(l\alpha_l^2)l}, 2 \neq l|D).$$

Here  $\alpha_l \in K^*$  are such that  $\mathfrak{a}_l = \mathfrak{p}_l \alpha_l$  is prime to  $\mathfrak{f}$  as in the proof of Lemma 3.3. Since

$$\prod_{l|\frac{D}{4}} \mathfrak{a}_l = \frac{D}{4} \prod_{l|\frac{D}{4}} \alpha_l$$

is principal, one has

$$(3.10) \quad \prod_{l|\frac{D}{4}} \sqrt{\epsilon_l(l\alpha_l^2)l} = \pm \prod_l \alpha_l^{-1} \chi\left(\prod_{l|\frac{D}{4}} \mathfrak{a}_l\right) \in L^*.$$

This implies that  $\sqrt{\epsilon_l(l\alpha_l^2)l}, 2 \neq l|D$  have a relation over  $L$ , and thus  $[T_g : K] = 2^r$ . So  $[T : K] = h$ .

For (3), notice first that  $3|D$  is necessary and  $\mu_2(\epsilon) = \{\pm 1\}$ . Notice also  $L = K(\epsilon) = K(\sqrt{-3})$ . If  $\epsilon(3\alpha_3^2) = 1$  (up to a cubic root of unity), then  $\sqrt{3} \in T_g$  by Lemma 3.3, and thus  $i = \sqrt{-1} \in T_g$ . This implies  $\dim A(\chi) = 2h$  by Theorem 3.4(2). If so  $\sqrt{3}$  can not be in  $T_g$ . If  $\epsilon(3\alpha_3^2) = -1$  (up to a cubic root of unity), then  $\sqrt{\epsilon(3\alpha_3^2)3} = \sqrt{-3} \in L$ . Then same argument as in (2) (in particular (3.10)) shows  $[T_g : K] = 2^r$  and thus  $\dim A(\chi) = h$ .

For (4), there is  $\beta \in K^*$  prime to  $D$  such that  $\epsilon(\beta)$  and  $\epsilon(\beta^2)$  has order 3. Let  $\alpha_2 \in K^*$  be such that  $2\alpha_2^2$  is prime to  $D$ . Then  $\epsilon(2\alpha_2^2)$  has order 4 if and only if  $\epsilon(2\alpha_2^2\beta^2)$  has order 12. The rest is similar to (2) and left to the reader.

**Corollary 3.6.** *When  $(6, D) = 1$ , every CM abelian variety over  $K$  of dimension  $h$  is  $K$ -isogenous to a scalar restriction of a CM elliptic curve over  $H$ .*

**Example 3.7.** Assume that  $4|D$  and  $D > 4$ . Let  $p \equiv 1 \pmod{4}$  be a prime number split or ramified in  $K = \mathbb{Q}(\sqrt{-D})$ . Let  $\mathfrak{p}$  be a prime ideal of  $K$  above  $p$ . Let  $\epsilon_p$  be a surjective character

$$\epsilon_p : (\mathcal{O}_K/\mathfrak{p})^* \cong (\mathbb{Z}/p)^* \longrightarrow \mu_4$$

so that a given generator  $g$  of  $(\mathbb{Z}/p)^*$  maps to  $i$ . Then

$$\epsilon_p(-1) = -1 \Leftrightarrow p \equiv 5 \pmod{8} \Leftrightarrow \epsilon_p(2) = \pm i.$$

Let  $f = \prod_p p$  be the product of finitely many such prime numbers  $p \equiv 1 \pmod{4}$ , and let  $\mathfrak{f}$  be an integral ideal of  $K$  whose norm is  $f$ . Let  $\epsilon = \prod_p \epsilon_p$  and assume that  $\epsilon(-1) = -1$ , i.e., odd number of  $p$ 's are congruent 5 modulo 8. Then  $\epsilon(2) = \pm i$  by the above equivalence. So any CM abelian variety over  $K$  of type  $\epsilon$  has dimension  $h$  by Theorem 3.5(2). On the other hand, let  $q \equiv 3 \pmod{4}$  be a prime split or ramified in  $K$  and let  $\mathfrak{q}$  be a prime above  $q$ . Let

$$\epsilon'_q : (\mathcal{O}_K/\mathfrak{q})^* \cong (\mathbb{Z}/q)^* \longrightarrow \{\pm 1\}$$

be such that  $\epsilon'_q(n) = \left(\frac{n}{q}\right)$  for an integer  $n$  prime to  $q$ . Then  $\epsilon'_q(-1) = -1$ . Now let  $f' = fq$  with  $f$  being as above but with even number of primes  $p \equiv 5 \pmod{8}$ . Let  $\epsilon' = \epsilon\epsilon'_q$ , then  $\epsilon'(-1) = -1$  is odd and  $\epsilon'(2) = \pm -1$ . So any CM abelian variety over  $K$  of type  $\epsilon'$  has dimension  $2h$ .

**Example 3.8.** Similarly, assume that  $3|D$ , and  $D > 3$ . Let  $p \equiv 1 \pmod{3}$  be an odd prime number split or ramified in  $K$  and let  $\epsilon_p$  be a surjective character

$$\epsilon_p : (\mathcal{O}_K/\mathfrak{p})^* \cong (\mathbb{Z}/p)^* \longrightarrow \mu_6.$$

Then

$$\epsilon_p(-1) = -1 \Leftrightarrow p \equiv 7 \pmod{12} \Leftrightarrow 3 \text{ is a square modulo } p.$$

Let  $f = \prod_p p$  be a product of odd primes  $p \equiv 1 \pmod{3}$  split or ramified in  $K$  and let  $\mathfrak{f}$  be an ideal of  $K$  whose norm is  $f$ . Let  $\epsilon = \prod_p \epsilon_p$  and assume that  $\epsilon(-1) = -1$ , i.e., there are odd number of prime divisors of  $f$  satisfying  $p \equiv 7 \pmod{12}$ . Then  $\epsilon(3)$  is of order 2 or 6. So any CM abelian variety over  $K$  of type  $\epsilon$  is of dimension  $h$ . On the other hand, if we let  $q \equiv -1 \pmod{12}$  and  $\epsilon'_q$  be as in Example 3.7. Let  $f$  and  $\epsilon$  be as above but with  $\epsilon(-1) = 1$ , then  $\epsilon' = \epsilon\epsilon'_q$  is odd and any CM abelian variety over  $K$  of type  $\epsilon'$  is of dimension  $2h$ .

**Example 3.9.** Similarly, assume  $12|D$ . Let  $p \equiv 1 \pmod{12}$  be a prime number split or ramified in  $K$ . Let  $\epsilon_p$  be a surjective character

$$\epsilon_p : (\mathcal{O}_K/\mathfrak{p})^* \cong (\mathbb{Z}/p)^* \longrightarrow \mu_{12}.$$

Define  $f$  and  $\epsilon$  the same way as in Example 3.7. If  $\epsilon(-1) = -1$ , then any CM abelian variety over  $K$  of type  $\epsilon$  is of dimension  $h$ . if  $\epsilon(-1) = 1$ , let  $q$  and  $\epsilon'_q$  be as in Example 3.7. Then any CM abelian variety over  $K$  of type  $\epsilon' = \epsilon\epsilon'_q$  is of dimension  $2h$ .

#### 4. Descent to $\mathbb{Q}$ .

Let  $\chi$  be a Hecke character of  $K$  satisfying (3.1) and let  $A(\chi)$  be an associated abelian variety over  $K$  of CM type  $(T, \Phi)$  with  $T = \mathbb{Q}(\chi)$  and  $\Phi$  being the set of complex embeddings of  $T$  which are the identity on  $K$ . If  $A(\chi)$  descends to an abelian variety over  $\mathbb{Q}$ , then (0.2) holds. For convenience, we repeat it here as

$$(4.1) \quad \chi(\bar{\mathfrak{a}}) = \overline{\chi(\mathfrak{a})}$$

for every ideal of  $K$  prime to the conductor  $\mathfrak{f}$  of  $\chi$ . Conversely, the proof of [Sh1, Proposition 5, Page 521] gives

**Lemma 4.1.** *Let  $\chi$  be a Hecke character of  $K$  satisfying (3.1) and (4.1). Then there is a CM abelian variety  $(A, i)$  over  $K$  of CM type  $(T, \Phi)$  associated to  $\chi$  such that  $(A, i_{T^+})$  is actually defined over  $\mathbb{Q}$ . Here  $T = \mathbb{Q}(\chi)$ ,  $\Phi$  is the set of embeddings of  $T$  which is the identity on  $K$ ,  $T^+$  is the maximal totally real subfield of  $T$  and  $i_{T^+}$  is the restriction of  $i$  on  $T^+$ .*

*Proof.* Choose  $\delta \in T^*$  such that  $\bar{\delta} = -\delta$ . Then there is, by [Sh1, Theorem 6, Page 512], a structure  $(A, \mathcal{C}, i)$  of type  $(T, \Phi, \mathcal{O}_T, \delta)$  rational over  $K$  which determines  $\chi$ . Here  $\mathcal{C}$  is a polarization of  $A$ . We refer to [Sh1, Page 509] for the meaning of type  $(T, \Phi, \mathcal{O}_T, \delta)$ . Let  $\rho$  be the complex conjugation of  $\mathbb{C}$ , restricting to  $T$  or  $K$ . Then [Sh1, Lemma 3, Page 520] asserts that  $(A^\rho, \mathcal{C}^\rho, i^*)$  is of the same type  $(T, \Phi, \mathcal{O}_T, \delta)$ , where  $i^*(a) = i(a^\rho)^\rho$ . So there is an isomorphism  $\mu$  from  $(A, \mathcal{C}, i)$  to  $(A^\rho, \mathcal{C}^\rho, i^*)$  over  $\mathbb{C}$ . On the other hand, (4.1) implies that they determine the same Hecke character  $\chi$  by [Sh1, Proposition 1, Page 511], and thus any isogeny between the two structures is defined over  $K$ . In particular, the isomorphism  $\mu$  is defined over  $K$ . Let

$$\omega : T_{\mathbb{R}} = T \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow A$$

be an homomorphism defining  $(A, \mathcal{C}, i)$  as of type  $(T, \Phi, \mathcal{O}_T, \delta)$ , and let  $\omega'(u) = \omega(\bar{u})^\rho$ . Then  $\omega'$  is an homomorphism from  $T_{\mathbb{R}}$  to  $A^\rho$  by [Sh1, Lemma 3, Page 520], and  $\mu$  can be chosen so that  $\mu' = \mu \circ \omega$ . So

$$\omega(u) = \omega'(\bar{u})^\rho = \mu^\rho(\omega(\bar{u})) = \mu^\rho \mu(\omega(u))$$

and thus  $\mu^\rho \mu = 1$ . By the descent theory,  $(A, \mathcal{C}, i_{T^+})$  can thus be descended to  $\mathbb{Q}$ .

By [Ro2, Proposition 1], the condition (4.1) is equivalent to each of the two following conditions

$$(4.2) \quad \epsilon(n) = \left( \frac{-D}{n} \right)$$

when  $n$  is prime to  $\mathfrak{f}$ , or

$$(4.3) \quad \chi^{un}|_{\mathbb{Q}_{\mathbb{A}}^*} = \kappa.$$

Here  $\kappa = \prod \kappa_p$  is the quadratic character of the ideles  $\mathbb{Q}_{\mathbb{A}}^*$  associated to  $K/\mathbb{Q}$  by the global class field theory, and  $\chi^{un} = \chi|_{\mathbb{A}^{\frac{1}{2}}}$  is the unitarization of  $\chi$ , viewed as an idele class character of  $K$ . In particular,  $\mathfrak{f}$  is divisible by every prime ideal of  $K$  which is ramified in  $K/\mathbb{Q}$ . In this section, we consider the set  $E$  of characters

$$(4.4) \quad \epsilon : (\mathcal{O}_K/\mathfrak{f})^* \longrightarrow \mu_{12}$$

satisfying (4.2) and that  $\mathfrak{f}$  is divisible only by ramified primes of  $K$ . For simplicity, we assume  $D > 4$  so  $\mathcal{O}_K^* = \{\pm 1\}$ . For  $\epsilon \in E$ , we will determine the dimension and conductor of a CM abelian variety over  $K$  of type  $\epsilon$ . Write

$$\mathfrak{f} = \prod_{p|D} \mathfrak{p}^{e_p}$$

then

$$(\mathcal{O}_K/\mathfrak{f})^* \cong \bigoplus_{p|D} (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{e_p})^*.$$

Here  $\mathfrak{p}$  is the unique prime ideal of  $K$  above  $p$ . So a character  $\epsilon$  satisfying (4.4) is the same as a character

$$(4.4') \quad \epsilon = \prod_{p|D} \epsilon_p : \prod_{p|D} \mathcal{O}_p^* \longrightarrow \mu_{12}$$

with  $\epsilon_p : \mathcal{O}_p^* \longrightarrow \mu_{12}$ . Here  $K_p = K \otimes \mathbb{Q}_p$  is the completion of  $K$  at the prime above  $p$ , and  $\mathcal{O}_p$  is the ring of integers in  $K_p$ . As remarked in [Ro2, Page 522],  $\chi_p^{un} = \epsilon_p^{-1}$  on  $\mathcal{O}_p^*$ , a fact we will use later. In particular, that  $\epsilon$  satisfies (4.2) is equivalent to that every  $\epsilon_p$  satisfies

$$(4.5) \quad \epsilon_p|_{\mathbb{Z}_p^*} = \kappa_p.$$

When  $p \nmid 6$ ,  $(\mathcal{O}_p)^* \cong (\mathcal{O}_p/\mathfrak{p})^* \times (1 + \varpi_p \mathcal{O}_p)$ , and  $(\mathcal{O}_p/\mathfrak{p})^* \cong (\mathbb{Z}/p)^*$ , where  $\varpi_p$  is uniformizer of  $\mathcal{O}_p$ . So there is a unique character  $\epsilon_p : \mathcal{O}_p^* \longrightarrow \mu_{12}$  satisfying (4.5), and it is trivial on  $1 + \varpi_p \mathcal{O}_p$ . We denote it by  $\epsilon_p^0$ . Similarly, when  $p = 3$ , there is a unique character  $\epsilon_3^0$  of  $\mathcal{O}_3^*$  of order 2 and conductor index 1 satisfying (4.5). Here the conductor index of a character  $\chi$  of  $K_p^*$  (or  $\mathcal{O}_p^*$ ) is the smallest integer  $r \geq 0$  such that  $\chi|_{1 + \varpi_p^r \mathcal{O}_p} = 1$ . Since  $1 + \varpi_3 \mathcal{O}_3$  is a cyclic pro-3-group, there are also two characters of  $\mathcal{O}_3^*$  of order 6 and conductor index 2 satisfying (4.5), given by  $\epsilon_3^0 \phi_3^{\pm 1}$ , where  $\phi_3$  is trivial on  $(\mathcal{O}_3/\mathfrak{p}_3)^* = \{\pm 1\}$  and of order 3 on  $1 + \varpi_3 \mathcal{O}_3$ . So the set  $E_3$  of characters of  $\mathcal{O}_3^*$  of order  $\leq 12$  satisfying (4.5) is

$$(4.6) \quad E_3 = \{\epsilon_3^0 \phi_3^i : -1 \leq i \leq 1\}.$$

The case  $p = 2$  is a little more complicated and interesting. Let  $G = \{z \in K_2 : z\bar{z} = 1\}$  be the norm one group, let

$$G_n = \{z = x + y\sqrt{-d} : y \in 2^n\mathbb{Z}_2, x - 1 \in 2^n d\mathbb{Z}_2\}$$

be subgroups of  $G$  for  $n \geq 0$ . Here  $d = D/4$ . Then  $G = G_0$  and  $[G_n : G_{n+1}] = 2$ . One has an exact sequence

$$1 \longrightarrow \mathbb{Z}_2^* \longrightarrow \mathcal{O}_2^* \longrightarrow G_1 \longrightarrow 1.$$

The last map is  $z \mapsto z/\bar{z}$ .

When  $8|D$ ,  $2|d$ , It is known ([Ro2, Proposition 4]) that  $\kappa_2$  extends to two quadratic characters  $\epsilon_2^\pm$  of  $\mathcal{O}_2^*$  of conductor index 5. Since  $G_1$  is a cyclic pro-2-group, the set  $E_2$  of characters of  $\mathcal{O}_2^*$  of order  $\leq 12$  satisfying (4.5) is

$$(4.7) \quad E_2 = \{\epsilon_2^\pm, \epsilon_2^\pm \tilde{\phi}\} = \{\epsilon_2^+ \tilde{\phi}^i : -1 \leq i \leq 2\}.$$

Here  $\phi$  is a fixed character of  $G_1$  of order 4, trivial on  $G_3$ , and  $\tilde{\phi}(z) = \phi(z/\bar{z})$ . Since

$$1 \longrightarrow (Z/8)^* \longrightarrow (\mathcal{O}_2/\varpi_2^5)^* \longrightarrow G_1/G_3 \longrightarrow 1,$$

all characters in  $E_2$  have conductor index 5.

When  $4||D$ ,  $\sqrt{-d}$  is a unit in  $\mathcal{O}_2$  and  $\varpi_2 = 1 + \sqrt{-d}$  is a uniformizer of  $K_2$ . In this case,  $G_1 = \{\pm 1\} \times G_2$  and  $G_2$  is a cyclic pro-2-group. It is easy to check that  $(\mathcal{O}_2/\varpi_2^3)^*$  is cyclic of order 4, generated by  $\delta = \sqrt{-d}$ . So

$$(4.8) \quad \epsilon_2^\pm : \mathcal{O}_2^* \longrightarrow (\mathcal{O}_2/\varpi_2^3)^* \longrightarrow \mu_4, \sqrt{-d} \mapsto \pm i$$

gives two extensions of  $\kappa_2$  to  $\mathcal{O}_2^*$ . Let  $\phi$  be a fixed character of  $G_1$  of order 4, trivial on  $\{\pm 1\} \times G_4$ . Then the set  $E_2$  of characters of  $\mathcal{O}_2^*$  of order  $\leq 12$  satisfying (4.5) is

$$(4.9) \quad E_2 = \{\epsilon^\pm \tilde{\phi}^i : -1 \leq i \leq 2\}.$$

The characters  $\epsilon_2^\pm \tilde{\phi}^i$  have conductor indices 3, 5, and 7 respectively according to  $i = 0, 2, \pm 1$ . So we have the following proposition.

**Proposition 4.2.** *Assume  $D > 4$ . Let  $E$  be the set of characters of  $(\mathcal{O}_K/\mathfrak{f})^*$  of order  $\leq 12$  satisfying (4.2) and that  $\mathfrak{f}$  is divisible only by primes dividing  $\sqrt{-D}\mathcal{O}_K$ . Then  $\#E = \#E_2 \cdot \#E_3$ , where  $E_p$  is the set defined in (4.6), (4.7), or (4.9) when  $p|D$  and trivial otherwise. Moreover*

(1) *When  $3|D$ ,  $\#E_3 = 3$ , one of which has conductor index 1, and the other two have conductor index 2.*

(2) When  $8|D$ ,  $\#E_2 = 4$ , all have conductor index 5.

(3) When  $4||D$ ,  $\#E_2 = 8$ , and their conductor indices are 3, 5, or 7.

(4) Let  $\epsilon = \prod_{p|D} \epsilon_p \in E$  and let  $\chi$  be a Hecke character of  $K$  of type  $\epsilon$ . The conductor of  $\chi$  is  $\prod_{p|D} \mathfrak{p}^{e_p}$  where  $e_p$  is the conductor index of  $\epsilon_p$ . One has  $e_p = 1$  for  $p \neq 6$ .

(5) Let  $\epsilon \in E$ , let  $\chi$  be a Hecke character of  $K$  of type  $\epsilon$ , and let  $A$  be a CM abelian variety over  $K$  associated with  $\chi$ . Then  $\dim A = h$  unless  $4|D$  and  $\epsilon_2 = \epsilon_2^\pm \tilde{\phi}$ , where  $\phi$  is a fixed character of  $G_1$  of order 4 given above. In such a case,

$$\dim A = \begin{cases} h & \text{if } d = D/4 \equiv 1 \pmod{8}, \\ 2h & \text{otherwise.} \end{cases}$$

*Proof.* Every claim except (5) was proved above. We use Theorem 3.5 to verify (5) in the case  $2|D$ , and leave the other cases to the reader.

We first assume  $8|D$  so that  $2||d$ . If  $\epsilon$  is quadratic, it is obvious by Theorem 3.5. If  $\epsilon_2 = \epsilon_2^\pm$  is quadratic, and  $\epsilon_3 = \epsilon_3^0 \phi_3 \in E_3$  is of order 6 as in (4.6). Let  $\alpha_3 = \frac{\beta}{\sqrt{-D}} \in K^*$  be such that  $3\alpha_3^2$  is prime to  $D$  and that  $\beta$  is prime to 3. Then

$$\begin{aligned} \epsilon(3\alpha_3^2) &= \phi_3(3\alpha_3^2) \epsilon_2^\pm(3) \epsilon_3^0(-3/D) \prod_{p|D, p \nmid 6} \epsilon_p^0(3) \\ &= \phi_3(3\alpha_3^2)(-3/D, -D)_3 \prod_{p \neq 3} (3, -D)_p \\ &= \phi_3(3\alpha_3^2) \prod_{p < \infty} (3, -D)_p (-1, -D)_3 \\ &= -\phi_3(3\alpha_3^2) \end{aligned}$$

generates  $\text{Im}(\epsilon)$ . So Theorem 3.5(3) implies that  $\dim A = h$ . When  $\epsilon_2 = \epsilon_2^\pm \tilde{\phi}$  is of order 4, let  $\alpha_2 = \frac{\beta}{\sqrt{-d}}$  such that  $2\alpha_2^2$  is prime to  $D$  and  $\beta$  is prime to 2. Then

$$\begin{aligned} \epsilon(2\alpha_2^2) &= \pm \epsilon_2(2\alpha_2^2) \prod_{2 \neq p|D} \epsilon_p(2\alpha_2^2) \\ &= \pm \kappa_2(-2/d) \prod_{2 \neq p|D} \epsilon_p(2\alpha_2^2) \end{aligned}$$

is of order 2 or 6. So Theorem 3.5 asserts that  $\dim A = 2h$ .

Finally, assume  $4||D$ , let  $\alpha = \frac{1+\sqrt{-d}}{2}\beta \in K^*$  be such that  $2\alpha_2^2 = (\frac{1-d}{2} + \sqrt{-d})\beta^2$  is prime to  $D$  and  $\beta$  is prime to 2. For  $\epsilon_2 = \epsilon_2^\pm \tilde{\phi}^j$  with  $-1 \leq j \leq 2$ .

Then

$$\begin{aligned}\epsilon(2\alpha_2^2) &= \pm\epsilon_2^\pm\left(\frac{1-d}{2} + \sqrt{-d}\right)\phi^j(g) \prod_{2 \neq p|D} \epsilon_p(2\alpha_2^2) \\ &= \pm i\phi^j(g) \prod_{2 \neq p|D} \epsilon_p(2\alpha_2^2)\end{aligned}$$

is of order 4 or 12 if and only if  $\phi^j(g) = \pm 1$ . Here

$$g = \frac{\sqrt{-d} + \frac{1-d}{2}}{\sqrt{-d} - \frac{1-d}{2}} \in G_2.$$

When  $j = 0$  or  $2$ , one has always  $\phi^j(g) = \pm 1$ . When  $j = \pm 1$ ,  $\phi^j(g) = \pm 1$  if and only if  $g \in G_3$ , which is in turn equivalent to  $d \equiv 1 \pmod{8}$ . This finishes the proof.

**Corollary 4.3.** *Assume  $D > 4$ . Let the notation be as in Proposition 4.2. Let  $E_{Sim}$  be the subset of characters  $\epsilon$  in  $E$  such that a CM abelian variety  $A$  over  $K$  of type  $\epsilon$  has dimension  $h$  and the smallest conductor. Then*

$$E_{Sim} = \begin{cases} \{\prod_{p|D} \epsilon_p^0\} & \text{if } 2 \nmid D, \\ \{\epsilon_2^\pm \prod_{2 \neq p|D} \epsilon_p^0\} & \text{if } 2|D. \end{cases}$$

In particular,  $\#E_{Sim} = 1$  or  $2$  depending on whether  $D$  is odd or even.

**Proof of Theorem 0.1** We may assume  $D > 4$ . By Remark 1.2, a simple CM abelian variety  $A$  over  $K$  is up to isogeny one-to-one correspondence to the set  $\{\chi^\sigma : \sigma \in \text{Aut}(\mathbb{C}/K)\}$ , where  $\chi$  is a Hecke character of  $K$  associated to  $(A, i)$  of CM type  $(T, \Phi)$  for some embedding  $i$  and some type  $(T, \Phi)$ . Here we require all the associated Hecke characters of  $K$  to be of infinite type  $\{\text{id}\}$ . When  $D$  is odd or divisible by 8,  $\epsilon$  is quadratic, and

$$\{\chi^\sigma : \sigma \in \text{Aut}(\mathbb{C}/K)\} = \{\chi\phi : \phi \text{ is an ideal class character of } K\}$$

is uniquely determined by  $\epsilon$ . So Theorem 0.2 follows from Corollary 4.3. Assume now that  $4||D$  and  $D > 4$ , and let  $L = K(i)$ . Fix  $\rho' \in \text{Aut}(\mathbb{C}/K)$  such that  $\rho'(i) = -i$ , and an ideal class character  $\phi'$  of  $K$  such that  $\phi'(\mathfrak{p}_2) = -1$  for the prime ideal  $\mathfrak{p}_2$  above 2. Fix an character  $\epsilon$  in Corollary 4.3 and a Hecke character  $\chi_0$  of  $K$  of type  $\epsilon$ . Then the other character in Corollary 4.3 is  $\epsilon^{-1}$ , and  $\chi'_0 = \chi_0^{\rho'}$  is of type  $\epsilon^{-1}$ . The Hecke characters of  $K$  satisfying (0.2) such that its associated abelian varieties have the smallest conductor and dimension

are of type  $\epsilon^{\pm 1}$  by Corollary 4.3 and are given by

$$\begin{aligned} & \{\chi_0\phi, \chi'_0\phi : \phi \text{ is an ideal class character of } K\} \\ &= \{\chi_0\phi, \chi'_0\phi : \phi \text{ is an ideal class character of } K \text{ such that } \phi(\mathfrak{p}_2) = 1\} \\ & \quad \bigcup \{\chi_0\phi'\phi, \chi'_0\phi'\phi : \phi \text{ is an ideal class character of } K \text{ such that } \phi(\mathfrak{p}_2) = 1\} \\ &= \{\chi_0^\sigma : \sigma \in \text{Aut}(\mathbb{C}/K)\} \bigcup \{(\chi_0\phi')^\sigma : \sigma \in \text{Aut}(\mathbb{C}/K)\}. \end{aligned}$$

Here we used the fact that

$$\begin{aligned} & \{\chi\phi : \phi \text{ is an ideal class character of } K \text{ such that } \phi(\mathfrak{p}_2) = 1\} \\ &= \{\chi^\sigma : \sigma \in \text{Aut}(\mathbb{C}/L)\}. \end{aligned}$$

Therefore, there are exactly two CM abelian varieties over  $K$  of dimension  $h$ , up to isogeny, which descend to  $\mathbb{Q}$  and have the smallest conductor.

From the proof, it is clear that the isogeny class of the CM abelian varieties  $A$  are not determined by  $\epsilon$  when the image of  $\epsilon$  is not in  $K$ .

## 5. Root number.

Let  $\chi$  be a simplest Hecke character of  $K$  of type  $\epsilon$ , given by Corollary 4.3. The condition (0.2) implies  $W(\chi) = \pm 1$ . It is known ([Ro3]) that

$$(5.0) \quad W(\chi) = \begin{cases} \left(\frac{2}{D}\right) & \text{if } 2 \nmid D, \\ \epsilon(1 + \sqrt{-D/4}) & \text{if } 8 \parallel D. \end{cases}$$

We may thus assume that  $4 \parallel D$  and  $D > 4$ . Let  $\mathfrak{p}_2$  be the prime ideal of  $K$  above 2,  $d = D/4$ , and  $\alpha_0 = 1 + \sqrt{-d}$ . Then  $\epsilon$  can be viewed as

$$(5.1) \quad \epsilon = \epsilon_2\epsilon^0 : (\mathcal{O}_K/\mathfrak{p}_2^3\sqrt{-d})^* = (\mathcal{O}_K/\mathfrak{p}_2^3)^* \times (\mathcal{O}_K/\sqrt{-d})^* \longrightarrow \mu_4$$

Here  $\epsilon_2$  is trivial on  $(\mathcal{O}_K/\sqrt{-d})^*$  but gives an isomorphism  $(\mathcal{O}_K/\mathfrak{p}_1^3)^* \cong \mu_4$ , while  $\epsilon^0$  is trivial on  $(\mathcal{O}_K/\mathfrak{p}_2^3)^*$  and is  $(\frac{2}{d})$  on  $(\mathcal{O}_K/\sqrt{-d})^* \cong (\mathbb{Z}/d)^*$ .

**Proposition 5.1.** *Let the notation be as above, and let  $\chi$  be a simplest Hecke character of  $K = \mathbb{Q}(\sqrt{-D})$  of type  $\epsilon$ . Then*

$$\chi(\alpha_0^{-1}\mathfrak{p}_2) = W(\chi)(1 - \epsilon_2(\sqrt{-d}))\alpha_0^{-1}.$$

*Proof.* For the proof, we follow [Ro2] closely. Let  $\kappa = \prod_p \kappa_p$  and  $\chi^{un} = \chi|_{\mathbb{A}^\times}^{\frac{1}{2}} = \prod_p \chi_p^{un}$  be as in as in Section 4. Then

$$(5.1ite) \quad \chi^{un}|_{\mathbb{Q}_\mathbb{A}^*} = \kappa, \quad \text{and} \quad \chi_p^{un}|_{\mathcal{O}_p^*} = \epsilon_p^{-1} \text{ for } p|D.$$

Here we write  $\epsilon = \prod_{p|D} \epsilon_p$  as in Corollary 4.3. Let  $\psi = \prod \psi_p$  be a nontrivial additive character of  $\mathbb{Q}_\mathbb{A}$  and let  $\psi_K = \psi \circ N_{K/\mathbb{Q}}$ . Then Rohrlich proved in [Ro2] that the relative local root number

$$W(\kappa_p, \chi_p) = \frac{W(\chi_p, \psi_{K_p})}{W(\kappa_p, \psi_p)}$$

is independent of the choice of  $\psi$ . Here  $W(\chi_p, \psi_{K_p})$  and  $W(\kappa_p, \psi_p)$  are local root numbers. Since the global root number of  $\kappa$  is one, we have then

$$(5.2) \quad W(\chi) = \prod_{p \leq \infty} W(\kappa_p, \chi_p).$$

By [Ro2, Propositions 8, 11, and 12], one has

$$(5.3) \quad W(\kappa_p, \chi_p) = \begin{cases} 1 & \text{if } p \nmid D, \\ \kappa_p(2) & \text{if } 2 \neq p|D. \end{cases}$$

So

$$(5.4) \quad W(\chi) = \prod_{2 \neq p|D} \kappa_p(2) W(\kappa_2, \chi_2) = \kappa_2(2) W(\kappa_2, \chi_2).$$

On the other hand, since  $\alpha_0^{-1} \mathfrak{p}_2$  is prime to  $D\mathcal{O}_K$ , one has

$$\begin{aligned} \chi(\alpha_0^{-1} \mathfrak{p}_2) &= \prod_{p \nmid D \infty} \chi_p(\alpha_0^{-1}) \\ &= \prod_{p|D \infty} \chi_p(\alpha_0) \\ &= \chi_2(\alpha_0) \alpha_0^{-1}. \end{aligned}$$

Combining this with (5.4) and the following lemma, one proves the proposition.

**Lemma 5.2.** *Let the notation be as above. Then*

$$W(\kappa_2, \chi_2) = \frac{1 + \chi_2(\sqrt{-d})}{\chi_2(1 + \sqrt{-d})} \kappa_2(2).$$

*Proof.* We first remark that  $\chi_2(\sqrt{-d}) = \epsilon_2(\sqrt{-d})^{-1} = -\epsilon_2(\sqrt{-d})$  by (5.1). We recall a general way to compute local root numbers. Let  $F$  be a non-archimedean local field with ring of integers  $\mathcal{O}_F$  and a uniformizer  $\varpi$ . Let  $\theta$  be a unitary

character of  $F^*$  of conductor index  $r$  and let  $\psi_F$  be a nontrivial additive character of  $F$  of conductor index  $n = n(\psi_F)$ , i.e., the smallest integer  $n$  such that  $\psi|_{\varpi^n \mathcal{O}_F} = 1$ . Then

$$(5.5) \quad W(\theta, \phi_F) = b^{-1} \theta^{-1}(\beta) \int_{\mathcal{O}_F^*} \theta^{-1}(x) \psi_F(x\beta) dx$$

where  $\beta \in \varpi^{-r+n} \mathcal{O}_F^*$  and  $b = |\varpi|^{\frac{r}{2}} \text{Vol}(\mathcal{O}_F, dx)$ . Now choose  $\psi_2 : \mathbb{Q}_2 \rightarrow \mathbb{C}^*$  via  $\psi_2(x) = e^{2\pi i \lambda_2(x)}$  where  $\lambda_2$  is the canonical map

$$\lambda_2 : \mathbb{Q}_2 \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

Then a simple calculation gives

$$(5.6) \quad W(\kappa_2, \psi_2) = \begin{cases} i & \text{if } 4 \parallel D, \\ i\kappa_2(2) & \text{if } 8 \parallel D \text{ and } \kappa_2(-1) = -1, \\ \kappa_2(2) & \text{if } 8 \parallel D \text{ and } \kappa_2(-1) = 1. \end{cases}$$

As for  $W(\chi_2, \psi_{K_2})$ , one has  $\chi_2^{un}(a) = \chi_2(a)|a|_2^{\frac{1}{2}}$ , and in particular,  $\chi_2^{un}(\alpha_0) = \chi_2(\alpha_0)/\sqrt{2}$ . Since the conductor index of  $\chi_2$  and  $\psi_{K_2}$  are 3 and  $-2$  respectively, and  $\alpha_0$  is a uniformizer of  $K_2$ , we may take  $\beta = \alpha_0/8$  in (5.5) and have

$$\begin{aligned} W(\chi_2, \psi_{K_2}) &= W(\chi_2^{un}, \psi_{K_2}) \\ &= \frac{2\sqrt{2}}{\text{Vol}(\mathcal{O}_2, dz)} \chi_2^{un}(\alpha_0/8)^{-1} \int_{\mathcal{O}_2^*} \chi_2^{un}(z^{-1}) \psi_{K_2}(z\alpha_0/8) dz \\ &= \frac{1}{2\sqrt{2}} \kappa_2(2) \chi_2^{un}(\alpha_0)^{-1} \sum_{a \in (\mathcal{O}_2/\mathfrak{p}_2^3)^*} \chi_2^{un}(a)^{-1} \psi_2\left(\frac{\text{tra}\alpha_0}{8}\right). \end{aligned}$$

Since  $(\mathcal{O}_2/\mathfrak{p}_2^3)^* = \{\pm 1, \pm\sqrt{-d}\}$ , one has

$$\begin{aligned} W(\chi_2, \psi_{K_2}) &= i\kappa_2(2) \chi_2^{un}(\alpha_0)^{-1} \frac{1 + \chi_2^{un}(\sqrt{-d})}{\sqrt{2}} \\ &= W(\kappa_2, \psi_2) \kappa_2(2) \frac{1 + \chi_2(\sqrt{-d})}{\chi_2(1 + \sqrt{-d})}. \end{aligned}$$

## 6. The central $L$ -value.

The purpose of this section is to prove the following theorem.

**Theorem 6.1.** *Let  $\chi$  be a simplest Hecke character of  $K = \mathbb{Q}(\sqrt{-D})$ . Then the central  $L$ -value  $L(1, \chi) \neq 0$  if and only if  $W(\chi) = 1$ .*

The case where  $D$  is odd or divisible by 8 was proved by Montgomery and Rohrlich ([MR]) in 1982. We will use the same method to settle the case  $4 \parallel D$  and  $D > 4$ . The case  $D = 4$  is well-known. We assume again in this section  $4 \parallel D$ . One complication is that the two family  $\{\chi^\sigma : \sigma \in C/K\}$  and  $\{\chi\phi : \phi \text{ is an ideal class character of } K\}$  are not the same in our case. Let  $L = K(i)$  and fix  $\rho' \in \text{Aut}(\mathbb{C}/K)$  such that  $\rho'(i) = -i$ . If  $\chi$  is of type  $\epsilon$  where  $\epsilon = \epsilon_2\epsilon^0$  is given by (5.1), then  $\chi' = \chi^{\rho'}$  is of type  $\epsilon^{-1} = \epsilon_2^{-1}\epsilon^0$ . For each ideal class  $C$  of  $K$ , let

$$(6.1) \quad L(s, C, \chi) = \sum_{\mathfrak{a} \in C, \text{ integral}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s}$$

be the partial  $L$ -function, and denote

$$(6.2) \quad \tilde{L}(s, C, \chi) = L(s, C, \chi) + L(s, C, \chi').$$

We also write  $[\mathfrak{a}]$  for the ideal class of  $K$  containing the ideal  $\mathfrak{a}$ . Since

$$\{\chi^\sigma : \sigma \in \text{Aut}(\mathbb{C}/L)\} = \{\chi\phi : \phi \text{ is an ideal class character of } K \text{ with } \phi([\mathfrak{p}_2]) = 1\},$$

one has by the same argument as in [Ro3, Page 226]

**Lemma 6.2.** *Let the notation be as above. Then the following are equivalent.*

- (1) *The central  $L$ -value  $L(1, \chi) = 0$ ,*
- (2) *The partial central  $L$ -values  $L(1, C, \chi) + L(1, C[\mathfrak{p}_2], \chi) = 0$  for every ideal class  $C$  of  $K$ .*
- (3) *The partial central  $L$ -values  $\tilde{L}(1, C, \chi) + \tilde{L}(1, C[\mathfrak{p}_2], \chi) = 0$  for every ideal class  $C$  of  $K$ .*

Following [MR], we recall that the theta function

$$(6.3) \quad \theta(t) = h + 2 \sum_{n \geq 1} a(n) e^{-\frac{2\pi n}{\sqrt{D}}}$$

is strictly increasing for  $t > 0$  and satisfies the simple functional equation

$$(6.4) \quad \theta(1/t) = t\theta(t).$$

Here  $a(n)$  is the number of integral ideals of  $K$  with norm  $n$ . Recall also that the Eisenstein series

$$(6.5) \quad G(z, s) = \sum_{n > 0, m \in \mathbb{Z}} \left( \frac{-D}{n} \right) (mD\bar{z} + n) |mDz + n|^{-2s}$$

has analytic continuation for all  $s \in \mathbb{C}$  and  $z$  in the upper half plane, and

$$\frac{\sqrt{D}}{\pi} G(z, 1) = h + 2 \sum_{n \geq 1} a(n) e^{2\pi n i z}.$$

In particular,

$$(6.6) \quad \frac{\sqrt{D}}{\pi} G\left(\frac{it}{\sqrt{D}}, 1\right) = \theta(t).$$

It is convenient to denote

$$G_{\text{odd}}(z, s) = G(z, s) - G(2z, s) = \sum_{m \text{ odd}, n > 0} \left(\frac{-D}{n}\right) (mD\bar{z} + n) |mDz + n|^{-2s}.$$

In particular,

$$(6.7) \quad \frac{\sqrt{D}}{\pi} G_{\text{odd}}\left(\frac{it}{\sqrt{D}}, 1\right) = \theta(t) - \theta(2t) > 0.$$

Since we can switch between  $\chi$  and  $\chi'$ , we may and will assume that  $\epsilon_2(\sqrt{-d}) = i$ .

**Lemma 6.2.** *Let  $\epsilon = \epsilon_2 \epsilon^0$  with  $\epsilon_2(\sqrt{-d}) = i$ .*

(1) *When  $\beta = m\sqrt{-d} + n$  is prime to  $D\mathcal{O}_K$ ,  $m, n \in \mathbb{Z}$ , one has*

$$\epsilon(\beta) = \begin{cases} \left(\frac{-D}{n}\right) & \text{if } m \equiv 0 \pmod{4}, \\ -\left(\frac{-D}{n}\right) & \text{if } m \equiv 2 \pmod{4}, \\ i \left(\frac{-1}{m}\right) \left(\frac{n}{d}\right) & \text{if } n \equiv 0 \pmod{4}, \\ -i \left(\frac{-1}{m}\right) \left(\frac{n}{d}\right) & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

(2) *When  $\beta = \alpha_0(a + b\frac{1-\sqrt{-d}}{2}) \in \alpha_0 \mathfrak{p}_2^{-1}$  is prime to  $D\mathcal{O}_K$ , where  $\alpha_0 = 1 + \sqrt{-d}$ ,  $a, b \in \mathbb{Z}$ , one has*

$$\epsilon(\beta) = \begin{cases} \left(\frac{-D}{2a+b}\right) & \text{if } a \equiv 0 \pmod{2}, \\ i \left(\frac{-D}{2a+b}\right) & \text{if } a \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* (1) follows from definition easily. For example, if  $m \equiv 2 \pmod{4}$ , then  $\beta = n - m + m\alpha_0 \equiv n - m \pmod{\mathfrak{p}_2^3}$  and thus

$$\epsilon(\beta) = \epsilon_2(\beta) \left(\frac{n}{d}\right) = \left(\frac{-1}{n-m}\right) \left(\frac{d}{n}\right) = -\left(\frac{-D}{n}\right).$$

(2) Notice that

$$\beta = a + \frac{1+d}{2}b + a\sqrt{-d}.$$

When  $a \equiv 0 \pmod{4}$ , one has

$$\begin{aligned} \epsilon(\beta) &= \left( \frac{-1}{a + \frac{1+d}{2}b} \right) \left( \frac{a + \frac{1+d}{2}b}{d} \right) \\ &= \left( \frac{2}{d} \right) \left( \frac{2a+b}{d} \right) \left( \frac{-1}{a + \frac{1+d}{2}b} \right) \\ &= \begin{cases} \left( \frac{d}{2a+b} \right) \left( \frac{-1}{b} \right) & \text{if } d \equiv 1 \pmod{8} \\ - \left( \frac{d}{2a+b} \right) \left( \frac{-1}{3b} \right) & \text{if } d \equiv 5 \pmod{8} \end{cases} \\ &= \left( \frac{-D}{2a+b} \right). \end{aligned}$$

The case  $a \equiv 2 \pmod{4}$  is similar. When  $a$  is odd, and  $d \equiv 1 \pmod{8}$ ,  $a + \frac{1+d}{2}b \equiv a + b \pmod{4}$ , and so

$$\begin{aligned} \epsilon(\beta) &= \begin{cases} i \left( \frac{-1}{a} \right) \left( \frac{2a+b}{d} \right) \left( \frac{2}{d} \right) & \text{if } a + b \equiv 0 \pmod{4}, \\ -i \left( \frac{-1}{a} \right) \left( \frac{2a+b}{d} \right) \left( \frac{2}{d} \right) & \text{if } a + b \equiv 2 \pmod{4} \end{cases} \\ &= i \left( \frac{-1}{2a+b} \right) \left( \frac{d}{2a+b} \right) \\ &= i \left( \frac{-D}{2a+b} \right). \end{aligned}$$

The case when  $a$  is odd and  $d \equiv 5 \pmod{8}$  is similar.

**Lemma 6.3.** *One has*

$$(6.8) \quad \tilde{L}(s, [\mathcal{O}_K], \chi) = 4G\left(\frac{2i}{\sqrt{D}}, s\right) - 2G\left(\frac{i}{\sqrt{D}}, s\right),$$

and

$$(6.9) \quad \tilde{L}(s, [\mathfrak{p}_2], \chi) = 2^s W(\chi) G_{\text{odd}}\left(\frac{i}{2\sqrt{D}}, s\right).$$

*Proof.* Since  $D > 4$ ,  $\mathcal{O}_K^* = \{\pm 1\}$ , one has by Lemma 6.2(1)

$$\begin{aligned}
\tilde{L}(s, [\mathcal{O}_K], \chi) &= \sum_{\alpha \in \mathcal{O}_K} (\chi(\alpha \mathcal{O}_K) + \chi'(\alpha \mathcal{O}_K))(N\alpha)^{-s} \\
&= \sum_{\alpha \in \mathcal{O}_K} \frac{\epsilon(\alpha) + \epsilon^{-1}(\alpha)}{2} \alpha (N\alpha)^{-s} \\
&= 2 \sum_{m \equiv 0(4), n > 0} \left( \frac{-D}{n} \right) (m\sqrt{-d} + n) |m\sqrt{-d} + n|^{-2s} \\
&\quad - 2 \sum_{m \equiv 2(4), n > 0} \left( \frac{-D}{n} \right) (m\sqrt{-d} + n) |m\sqrt{-d} + n|^{-2s} \\
&= 2G\left(\frac{2i}{\sqrt{D}}, s\right) - 2G_{\text{odd}}\left(\frac{i}{\sqrt{D}}, s\right) \\
&= 4G\left(\frac{2i}{\sqrt{D}}, s\right) - 2G\left(\frac{i}{\sqrt{D}}, s\right).
\end{aligned}$$

This proves (6.8). On the other hand,  $\mathfrak{a} \in [\mathfrak{p}_2]$  is integral and prime to  $D\mathcal{O}_K$  if and only if  $\mathfrak{a} = \beta\alpha_0^{-1}\mathfrak{p}_2$  with  $\beta \in \alpha_0\mathfrak{p}_2^{-1}$  prime to  $D\mathcal{O}_K$ . In such a case, Proposition 5.1 gives

$$\begin{aligned}
\chi(\mathfrak{a}) &= \epsilon(\beta)\beta\chi(\alpha_0^{-1}\mathfrak{p}_2) \\
&= W(\chi)(1 - \epsilon_2(\sqrt{-d}))\alpha_0^{-1}\epsilon(\beta)\beta.
\end{aligned}$$

One also has  $W(\chi) = W(\chi')$  by Proposition 5.1. So

$$\begin{aligned}
\chi(\mathfrak{a}) + \chi'(\mathfrak{a}) &= W(\chi)\alpha_0^{-1}\beta \left[ \epsilon(\beta)(1 - \epsilon_2(\sqrt{-d})) + \epsilon(\beta)^{-1}(1 + \epsilon_2(\sqrt{-d})) \right] \\
&= W(\chi)2\alpha_0^{-1}\beta \begin{cases} \epsilon(\beta) & \text{if } \epsilon(\beta) = \pm 1, \\ -\epsilon(\beta)\epsilon_2(\sqrt{-d}) & \text{if } \epsilon(\beta) = \pm i. \end{cases} \\
&= W(\chi)(2a + b - b\sqrt{-d}) \left( \frac{-D}{2a + b} \right),
\end{aligned}$$

if  $\beta = \alpha_0(a + b\frac{1-\sqrt{-d}}{2}) \in \alpha_2\mathfrak{p}_2^{-1}$  as in Lemma 6.2, and  $\epsilon_2(\sqrt{-d}) = i$  (our assumption). So

$$\begin{aligned}
\tilde{L}(s, [\mathfrak{p}_2], \chi) &= \sum_{\mathfrak{a} \in [\mathfrak{p}_2], \text{ integral}} (\chi(\mathfrak{a}) + \chi'(\mathfrak{a}))(N\mathfrak{a})^{-s} \\
&= W(\chi)2^{s-1} \sum_{b \text{ odd}, a \in \mathbb{Z}} \left( \frac{-D}{2a + b} \right) (2a + b - b\sqrt{-d}) |2a + b - b\sqrt{-d}|^{-2s}.
\end{aligned}$$

Substituting  $n = 2a + b$  and  $m = n$ , one gets

$$\begin{aligned}\tilde{L}(s, [\mathfrak{p}_2], \chi) &= W(\chi)2^{s-1} \sum_{m, n \text{ odd}} \left(\frac{-D}{n}\right) (n - m\sqrt{-d})|n - m\sqrt{-d}|^{-2s} \\ &= W(\chi)2^s G_{\text{odd}}\left(\frac{i}{2\sqrt{D}}, s\right).\end{aligned}$$

**Proof of Theorem 6.1.** Now it is easy to verify Theorem 6.1. Indeed, Lemma 6.3 gives

$$\begin{aligned}\tilde{L}(1, [\mathcal{O}_K], \chi) + \tilde{L}(1, [\mathfrak{p}_2], \chi) &= 4G\left(\frac{2i}{\sqrt{D}}, 1\right) - 2G\left(\frac{i}{\sqrt{D}}, 1\right) + 2W(\chi)G_{\text{odd}}\left(\frac{i}{2\sqrt{D}}, 1\right) \\ &= \frac{\pi}{\sqrt{D}}(4\theta(2) - 2\theta(1)) + 2W(\chi)\frac{\pi}{\sqrt{D}}\left(\theta\left(\frac{1}{2}\right) - \theta(1)\right) \\ &= \frac{2\pi}{\sqrt{D}}(1 + W(\chi))\left(\theta\left(\frac{1}{2}\right) - \theta(1)\right).\end{aligned}$$

Since  $\theta(\frac{1}{2}) - \theta(1) > 0$ , Lemma 6.2 gives Theorem 6.1.

## 7. The Central Derivative.

Let  $\chi$  be a simplest Hecke character of  $K = \mathbb{Q}(\sqrt{-D})$  of conductor  $\mathfrak{f}$ , and let

$$(7.1) \quad \Lambda(s, \chi) = \left(\frac{B}{2\pi}\right)^s \Gamma(s)L(s, \chi),$$

where

$$(7.2) \quad B = \begin{cases} D & \text{if } 2 \nmid D, \\ \sqrt{2}D & \text{if } 4 \parallel D, \\ 2D & \text{if } 8 \parallel D. \end{cases}$$

Then  $\Lambda(s, \chi)$  has analytic continuation to the whole complex  $s$ -plane, and satisfies the functional equation

$$(7.3) \quad \Lambda(s, \chi) = W(\chi)\Lambda(2 - s, \chi).$$

Here  $W(\chi) = \pm 1$  is the root number of  $\chi$ . In section 6, we proved that  $\Lambda(1, \chi) \neq 0$  if and only if  $W(\chi) = 1$ . In this section, we prove

**Theorem 7.1.** *Let  $\chi$  be a simplest Hecke character of  $K$ . Then  $\Lambda'(1, \chi) \neq 0$  if and only if  $W(\chi) = -1$ .*

*Proof.* The case where  $D$  is odd or divisible by 8 was proved by S. Miller and the author ([MY]). We will settle the case  $4 \parallel D$  and  $D > 4$  the same way. The case  $D = 4$  does not occur here. One direction is trivial by the functional equation. Now assume  $W(\chi) = -1$ . For simplicity, set

$$(7.4) \quad L(s) = \tilde{L}(s, [\mathcal{O}_K], \chi) + \tilde{L}(s, [\mathfrak{p}_2], \chi), \quad \Lambda(s) = \left(\frac{B}{2\pi}\right)^s \Gamma(s)L(s).$$

Then

$$(7.5) \quad \Lambda(s) = W(\chi)\Lambda(2-s) = -\Lambda(2-s).$$

The same argument as in the proof of [MY, Lemma 2.1] gives

**Lemma 7.2.** *When  $W(\chi) = -1$  and  $\Lambda'(1) \neq 0$ , one has  $\Lambda'(1, \chi) \neq 0$ .*

So to prove the theorem, it suffices to show  $\Lambda'(1) \neq 0$  when  $W(\chi) = -1$ . By Cauchy's theorem and (7.5), one has

$$(7.6) \quad \begin{aligned} \Lambda'(1) &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Lambda(s) \frac{ds}{(s-1)^2} - \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Lambda(s) \frac{ds}{(s-1)^2} \\ &= \frac{2}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Lambda(s) \frac{ds}{(s-1)^2}. \end{aligned}$$

The same calculation as in the proof of Lemma 6.3 gives

$$\begin{aligned} \tilde{L}(s, [\mathcal{O}_K], \chi) &= \sum_{\substack{v \text{ even}, \\ u \text{ odd}}} \epsilon(u + v\sqrt{-d})(u + v\sqrt{-d})|u^2 + v^2d|^{-s} \\ &= 2 \sum_{\substack{u > 0, \\ \text{odd}}} \left(\frac{-D}{u}\right) u^{1-2s} + 2 \sum_{\substack{v > 0, \text{even} \\ u > 0, \text{odd}}} \epsilon(u + v\sqrt{-d})2u|u^2 + v^2d|^{-s} \end{aligned}$$

So

$$(7.7) \quad \frac{1}{2} \tilde{L}(s, [\mathcal{O}_K], \chi) = \sum_{n > 0} \left(\frac{-D}{n}\right) n^{1-2s} + \sum_{n > 0} a(n)n^{-s}$$

with

$$(7.8) \quad a(n) = \sum_{\substack{u^2 + v^2d = n \\ u > 0, \text{odd}, v > 0, \text{even}}} \epsilon(u + v\sqrt{-d})2u.$$

Similarly, Lemma 6.3(2) gives

$$(7.9) \quad \frac{1}{2} \tilde{L}(s, [\mathfrak{p}_2], \chi) = - \sum_{n>0} b(n) (n/2)^{-s},$$

with

$$(7.10) \quad b(n) = \sum_{\substack{u^2+v^2d=n \\ u>0, v>0, \text{ odd}}} \left( \frac{-D}{u} \right) 2u.$$

Following [MY, pages 265-266], let

$$(7.11) \quad f(x) = \frac{\Gamma(0, x)}{x} = \frac{1}{x} \int_x^\infty e^{-t} \frac{dt}{t}$$

be the inverse Mellin transform of  $\frac{\Gamma(s)}{(s-1)^2}$ , then

$$(7.12) \quad \frac{1}{4} \Lambda'(1) = R + C_1 - C_2$$

where

$$\begin{aligned} R &= \sum_{n>0} \left( \frac{-D}{n} \right) n f\left(\frac{2\pi n^2}{B}\right), \\ C_1 &= \sum_{n>0} a(n) f\left(\frac{2\pi n}{B}\right), \\ C_2 &= \sum_{n>0} b(n) f\left(\frac{\pi n}{B}\right). \end{aligned}$$

[MY, (2.12)] asserts that the main term is

$$(7.13) \quad R > .5235B - .8458B^{\frac{3}{4}} - .3951B^{\frac{1}{2}}.$$

The same argument as in [MY, Section 4] gives the trivial upper bounds for  $C_1$  and  $C_2$  as follows:

$$(7.14) \quad \begin{aligned} |C_1| &\leq \frac{1}{4\pi^2} \sum_{v>0} v^{-4} e^{-2\sqrt{2}\pi v^2} \cdot \sum_{u>0} u e^{-\frac{2\pi}{B}u^2} \\ &\leq \frac{1}{4\pi^2} \times 0.0001383 \times \frac{B}{4\pi} \approx 2.789 \times 10^{-7} B \end{aligned}$$

and

$$\begin{aligned}
 |C_2| &\leq \frac{16}{\pi^2} \sum_{v>0, \text{ odd}} v^{-4} e^{-\frac{\pi v^2}{2\sqrt{2}}} \sum_{u>0} u e^{-\frac{\pi u^2}{B}} \\
 (7.15) \quad &\leq \frac{16}{\pi^2} \times 0.3293220848 \frac{B}{2\pi} \approx .0850B.
 \end{aligned}$$

Combining (7.12) – 7.15), one has

$$\frac{1}{4} \Lambda'(1) \geq .4385B - .8458B^{\frac{3}{4}} - .3951B^{\frac{1}{2}} > 0$$

when  $B = \sqrt{2}D \geq 42\sqrt{2}$ . This leaves one exception  $D = 20$  ( $D = 4$  does not occur here). A simple numerical calculation shows  $\Lambda'(1) > 0$  in this case too. Now Theorem 7.1 follows from Lemma 7.2.

We remark that a little more work as in [MY] and [LX] would show that the beautiful formula (0.4) also holds for ‘small’ quadratic twists of the simplest Hecke characters. Similar results should also hold for odd powers of these characters.

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