

**HEEGNER ZEROS OF THETA FUNCTIONS**  
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ABSTRACT. Heegner divisors play an important role in number theory. However little is known on whether a modular form has Heegner zeros. In this paper, we start to study this question for a family of classical theta functions, and prove a quantitative result, which roughly says that many of these theta functions have a Heegner zero of discriminant  $-7$ . This leads to some interesting questions on the arithmetic of certain elliptic curves, which we also address here.

**0. Introduction.**

Let  $N \geq 1$  be an integer and let  $f$  be a non-zero meromorphic modular form of level  $N$  with algebraic Fourier coefficients. Then  $f$  can be viewed as a (meromorphic) section of a line bundle on the modular curve  $X_0(N)$  and thus its zeros and poles give a divisor in  $X_0(N)$  which is algebraic. These important divisors appear in the beautiful works of Rohrlich ([R]) on Jensen's formula and more recently of Bruinier, Kohnen, and Ono ([B-K-O]) on the values of modular functions. However, if we let  $\tau$  be a zero or a pole of  $f$  on the upper half plane  $\mathbb{H}$ , then it is well-known that  $\tau$  is either quadratic (Heegner point) or transcendental. So it is very interesting to isolate and understand the Heegner zeros/poles of  $f$ . We recall that a Heegner point on  $X_0(N)$  of discriminant  $-D$  is represented by a quadratic number  $\tau = \frac{b+\sqrt{-D}}{2aN}$  with integers  $a > 0$  and  $b$ .

Although Heegner points play very important roles in many branches of number theory, such as the Gross-Zagier formula, Kolyvagin's Euler system, and the Borcherds product theory, to name a few, little is known about the Heegner zeros of modular forms.

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In this paper, we study the Heegner zeros for a family of classical theta functions

$$(0.1) \quad \theta_d(z) = \sum_{(x,d)=1} \left(\frac{d}{x}\right) e(x^2 z),$$

where  $d \equiv 1 \pmod{4}$  is a square-free integer and  $e(z) = e^{2\pi iz}$ . It is a modular form for  $\Gamma_0(4d^2)$  of weight  $\frac{1}{2}$ .

When  $d = 1$ , the classical theta function has no zeros in the upper half plane. When  $d = 5$ , it is proved in [Y, Proposition 3.8] that  $\theta_5(z)$  does not vanish at any Heegner points of  $X_0(100)$  of any fundamental discriminant. In general, for a fixed  $d$ , there are obviously only finitely many  $D$  such that  $\theta_d$  vanishes at a Heegner point of  $X_0(4d^2)$  of discriminant  $-D$ .

On the other hand, for a fixed  $D$  one may ask if there are infinitely many twisted theta function  $\theta_d(z)$  vanishing at a Heegner point of  $X_0(4d^2)$  of discriminant  $-D$ . We first note that  $X_0(4d^2)$  has a Heegner point of discriminant  $-D$  if and only if every prime factor of  $2d$  splits in  $K_D = \mathbb{Q}(\sqrt{-D})$  and so one has to have  $D \equiv 7 \pmod{8}$ . In this paper we will settle the case where  $D = 7$ .

**Theorem 0.1.** *Let  $N(X)$  be the set of positive square free integers  $d \equiv 1 \pmod{4}$ ,  $d \leq X$  such that every prime factor of  $d$  splits in  $\mathbb{Q}(\sqrt{-7})$  and that the Heegner point  $\tau_d$  of  $X_0(4d^2)$  with discriminant  $-7$  is a zero of  $\theta_d$ . Then*

$$|N(X)| \gg X^{1/3} / \log X.$$

The proof is based on the relation given by F. Rodriguez Villegas and T. Yang in [RV-Y] between Heegner zeros of twisted theta functions  $\theta_d(z)$  and the arithmetic of a precise family of CM elliptic curves  $A(D)$  constructed by B. Gross in [G1]. (See section 3 for a brief summary). This relation allows us to restate the problem in terms of zeros of Hasse-Weil  $L$ -functions. In particular, let  $A(D)$  be the elliptic curve constructed in [G1]. For any  $d > 1$  let  $A(D)^d$  be the  $d$ -quadratic twist of  $A(D)$ , and let  $L(s, A(D)^d)$  be its Hasse-Weil  $L$ -function over its definition field  $F_D = \mathbb{Q}(j)$  with  $j = j(\frac{1+\sqrt{-D}}{2})$ . Corollary 3.5 in [RV-Y] is the following

**Theorem A.** *Assume  $d > 1$  and  $D \equiv 7 \pmod{8}$ . If all the prime factors of  $d$  split in  $K_D$ , then the following are equivalent.*

- i) *The theta function  $\theta_d$  vanishes at one (and all) the Heegner points of  $X_0(4d^2)$  with discriminant  $-D$ .*
- ii)  $L(1, A(D)^d) = 0$ .

On the other hand, a celebrated theorem of Kolyvagin and Logachev ([K-L]) gives in our case that  $L(1, A(D)^d) = 0$  whenever  $A(D)^d$  has positive rank. Therefore, the proof of Theorem 0.1 is reduced to the following theorem

**Theorem 0.2.** *Let  $N(X)$  be the set of positive square free integers  $d \equiv 1 \pmod{4}$ ,  $d \leq X$  such that  $A(7)^d(\mathbb{Q})$  has a point of infinite order, and that every prime factor of  $d$  splits in  $\mathbb{Q}(\sqrt{-7})$ . Then*

$$|N(X)| \gg X^{1/3} / \log X.$$

This kind of problem has already been considered by many authors ([G-M], [S-T],[ J]), where different lower bounds are given on the number of  $d$  such that the quadratic twist  $E^d(\mathbb{Q})$  has positive rank for any elliptic curve  $E$  over  $\mathbb{Q}$ . We will now use this type of technique for  $A(7)^d$  with the extra condition that every prime factor of  $d$  is split in  $\mathbb{Q}(\sqrt{-7})$ . We will use polynomial twists  $d = d(t)$  which arise naturally from the Weierstrass equation of the elliptic curve.

A Weierstrass equation for  $A(7)$  is already given by B. Gross in [G2]

$$(0.2) \quad A(7) : y^2 + xy = x^3 - x^2 - 2x - 1.$$

In fact in [G2] a minimal model (a Weierstrass equation with minimal discriminant) is given for any  $A(D)$  whenever  $D = p$  is a prime, although in general it is defined over the number field  $F_D$ . However, there is no known minimal model for  $A(D)$  for composite  $D$ . This raises two interesting questions in order to extend Theorem 0.1 for a general discriminant  $-D$ . First of all

**Question 0.3** Is there always a minimal model of  $A(D)$  for composite  $D$ ? How to construct it if it exists?

A constructive answer to these questions is expected when  $D$  is relatively prime to 6. Furthermore, for a fixed  $D$  we need pairs  $(d, P)$  where  $d$  is rational and  $P$  is of infinite order in  $A(D)^d$ , and so

**Question 0.4** Given  $D > 7$ , are there infinitely many square free integers  $d > 0$  prime to  $D$  such that  $A(D)^d(F_D)$  is infinite?

More generally, given an elliptic curve  $E$  over a number field  $F$  which does not descend to  $\mathbb{Q}$ , are there always infinitely many non-equivalent rational quadratic twists  $E^d$  which has an  $F$ -point of infinite order, subject to some root number condition?

We will give an answer to both questions for  $D = 15$  in section 3.

Another natural way to extend Theorem 0.1 is to study the arithmetic of  $d$ . In particular, one may ask the following

**Question 0.5** Are there infinitely many primes  $p$  such that  $\theta_p$  vanishes at a Heegner point of  $X_0(4p^2)$  of discriminant  $-7$ ?

An affirmative answer would follow from a general conjecture about the rank of prime twists (see [J]). In section 2, using a weighted sieve inequality in a similar way as in [J], we will prove the following theorem. Let us write  $d = P_r$  if the number of primes dividing  $d$ , counting multiplicities, is bounded by  $r$ .

**Theorem 0.6.** *Let  $N_r(X)$  be the  $P_r$  elements in  $N(X)$ . Then*

$$|N_6(X)| \gg X^{1/3} / \log^2 X.$$

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## 1. Proof of Theorem 0.2.

To clarify the exposition let us make the following definition:

**Definition.** *A positive integer  $d$  is “good” if  $d \equiv 1 \pmod{4}$  is squarefree with only prime factors splitting in  $\mathbb{Q}(\sqrt{-7})$  and such that  $A(7)^d(\mathbb{Q})$  has a point of infinite order.*

For convenience, let  $E = A(7)$  be as in (0.2). By change of variables  $16x \mapsto 28x + 1$  and  $64y \mapsto y + 56x + 2$  in (0.2), we find

$$(1.1) \quad E : \quad y^2 = (28x - 31)((28x + 11)^2 + 28) = p(x)F(x).$$

For any integer  $t > 1$  let us write  $p(t)F(t) = d(t)B(t)^2$  for  $d(t)$  squarefree and  $B(t)$  a positive integer. We will consider the twist  $E^{d(t)}$  together with the point  $(t, B(t)) \in E^{d(t)}(\mathbb{Q})$ . For these twists we have

**Lemma 1.1.** *Let the notation be as above. If  $p(t) = 28t - 31$  is prime, then  $d(t)$  is good and the root number of  $E^{d(t)}$  is  $+1$ , with at most a finite number of exceptions.*

*Proof.* Clearly  $p(t) \equiv F(t) \equiv 1 \pmod{4}$ . So  $B(t)$  is odd and  $B(t)^2 \equiv 1 \pmod{4}$ , and thus  $d(t) \equiv 1 \pmod{4}$ . Next, for every prime  $l|d(t)$ , either  $l = p(t)$  or  $F(t) \equiv 0 \pmod{l}$ . When  $l = p(t)$ , one has  $\left(\frac{l}{7}\right) = \left(\frac{-31}{7}\right) = 1$ , and so  $l$  is split in  $\mathbb{Q}(\sqrt{-7})$ . When

$$F(t) = (28t + 11)^2 + 28 \equiv 0 \pmod{l},$$

one sees that  $-7$  is a square modulo  $l$ . So  $l$  is again split in  $\mathbb{Q}(\sqrt{-7})$ , and thus every prime factor of  $d(t)$  is split in  $\mathbb{Q}(\sqrt{-7})$ .

On the other hand, in [G-M] the authors proved that for all but finitely many  $t$ , the point  $(t, B(t))$  of  $E^{d(t)}(\mathbb{Q})$  has infinite order. Finally,  $d(t) > 0$  implies that  $E^{d(t)}$  has root number 1, (see [G1, Cor 19.2.8]).

**Lemma 1.2.** *Let the notation be as above. For any integer  $d$ , let  $P(d)$  be the set of primes  $p = 28t - 31 \in (T^{1/2}, T)$  such that  $d(t) = d$ . Then  $|P(d)| \leq 5$  for  $T \gg 1$ .*

*Proof.* For  $0 \leq i \leq r$ , let  $t_i$  be such that  $p(t_i) \in P(d)$ . Noting that

$$F(x) = (28x + 53)(28x - 31) + 2^8 \times 7,$$

we see that  $(F(t), p(t)) = 1$  for any integer  $t$ . In particular it immediately follows that  $p(t_i) | d$  for  $0 \leq i \leq r$ . Hence,

$$F(t_0) = B(t_0)^2(d/p(t_0)) = B(t_0)^2 \frac{d}{\prod_{i=0}^r p(t_i)} \prod_{i=1}^r p(t_i) \geq B(t_0)^2 \prod_{i=1}^r p(t_i) \geq T^{r/2},$$

since  $p > T^{1/2}$  for any  $p \in P(d)$ .

On the other hand  $t_0 \leq T$ , so  $F(t_0) \ll T^2$  and thus  $r \leq 4$ , which completes the proof of the lemma.

**Proof of Theorem 0.2:** For  $T \gg 1$ , let  $X = p(T)F(T) \asymp T^3$ . Lemmas 1.1 and 1.2 allow us to establish the lower bound

$$|N(X)| \gg |\{T^{1/2} < p < T : p \equiv -31 \pmod{28} \text{ is prime}\}|.$$

Theorem 0.2 now follows easily from the Prime Number Theorem in arithmetic progression.

*Remark 1.3.* The proof of Theorem 0.2 explicitly constructs a twist  $d$  and a point of infinite order in  $A(7)^d(\mathbb{Q})$ . In practice, it is possible to take  $p(t)$  to be composite. For example, choosing  $t = 2$ , we find  $p(t) = 5^2$  and direct computation using the same procedure gives that  $A(7)^{4517}$  has the point  $(57/16, 119/64)$  of infinite order. Thus  $L(1, \chi_{7,4517}) = 0$  and so  $\theta_{4517}$  vanishes at the Heegner point of  $X_0(4 \cdot 4517^2)$  with discriminant  $-7$ . Incidentally, the smallest quadratic twist of  $A(7)$  to have a point of infinite order is  $A(7)^{53}$ .

## 2. Proof of Theorem 0.6.

To prove Theorem 0.6 we will bound the number of prime factors of the good integers found in Theorem 0.2. For this purpose we will use a linear weighted sieve inequality for the polynomial  $p(x)F(x)$  defined in the previous section.

In particular we will use a direct application of Theorem 9.3 in [H-R, page 253]. For completeness, we include the hypotheses and state a particular case of this theorem which we shall refer to as Theorem T1.

Let  $X \gg 1$ , and consider a set  $A$  of integers in  $[1, X]$ . We denote  $A_d = \{a \in A : d|a\}$ . Suppose that for any square-free  $d$  we can write

$$(2.0) \quad |A_d| = X \frac{\omega(d)}{d} + R_d$$

for some multiplicative function  $\omega(d)$  and a function  $R_d$ , satisfying the following conditions:

$$(\Omega_1) \quad 1 \leq \frac{1}{1 - \omega(p)/p} \leq C_1, \text{ for any prime } p,$$

$$(\Omega_2^*(1)) \quad -C_2 \log \log 3X \leq \sum_{v \leq p < w} \frac{\omega(p)}{p} \log p - \log \frac{w}{v} \leq C_2, \quad 2 \leq v \leq w,$$

$$(\Omega_3) \quad \sum_{z \leq p < y} |A_{p^2}| \leq C_3 \left( \frac{X \log X}{z} + y \right), \quad 2 \leq z \leq y,$$

$$(R(1, \alpha)) \quad \sum_{d < X^\alpha / (\log X)^{C_4}} \mu^2(d) 3^{\nu(d)} |R_d| \leq C_5 \frac{X}{\log^2 X}, \quad X \geq 2,$$

for absolute constants  $C_1, C_2, C_3, C_4, C_5$  and  $\alpha > 0$ . Then we have

**Theorem T1.** *Let  $A$  be a set satisfying the above conditions (2.0),  $(\Omega_1)$ ,  $(\Omega_2^*(1))$ ,  $(\Omega_3)$ , and  $(R(1, \alpha))$ . Assume further that  $|a| < X^{\alpha(r-1)+\varepsilon}$  for all  $a \in A$  and some  $\varepsilon > 0$ . Then there exists a constant  $X_0(r) > 0$  such that*

$$|\{a \in A : a = P_r\}| \geq \frac{1}{7\alpha} \prod_p \frac{1 - \omega(p)/p}{1 - 1/p} \frac{X}{\log X},$$

for  $X \geq X_0(r)$ . Here  $a = P_r$  whenever  $a$  has at most  $r$  prime factors, counting multiplicity, as in Theorem 0.6.

Let  $p(x)$  and  $F(x)$  be defined as in the previous section. In order to prove Theorem 0.6 for a given large  $X$ , we have to apply Theorem T1 to the sequence  $\{p(t)F(t) < X : p(t) \text{ prime}\}$ . This can be written as

$$(2.1) \quad \{pG(p) < X : p \equiv -31 \pmod{28}\},$$

where  $G(x) = F((x + 31)/28) = x^2 + 84x + 1792$ .

We will deduce Theorem 0.6 from the application of Theorem T1 to the general sequence of polynomials evaluated at primes

$$(2.2) \quad A = A(f, k, l) = \{f(p) : p \leq x \text{ prime}, p \equiv l \pmod{k}\},$$

for a pair of fixed integers  $(k, l) = 1$  and an irreducible polynomial  $f(x) \in \mathbb{Z}[x]$ . Hence, our first goal is to verify that (2.2) satisfies the hypotheses of Theorem T1.

The sequence in (2.2), given as Example 6 in [H-R, page 22], is a generalization of that in Theorem 9.8 [H-R, page 261] with the additional condition that the primes are in a certain congruence class. As in [H-R], to verify that the sequence (2.2) satisfies the conditions in Theorem T1, we first prove a series of technical lemmas.

For any given integer  $q$ , let

$$E(x, q) = \max_{2 \leq y \leq x} \max_{\substack{1 \leq h \leq q \\ (h, q) = 1}} \left| \pi(y; h, q) - \frac{\text{li}(y)}{\varphi(q)} \right|,$$

where  $\pi(y; h, q)$  is the number of primes congruent to  $h$  modulo  $q$  and less than  $y$ , and

$$\text{li}(y) = \int_2^y \frac{dt}{\log t}$$

is the usual logarithmic integral function  $li$  which is asymptotic to  $\frac{y}{\log y}$  as  $y$  goes to infinity.

**Lemma 2.1.** *Let  $h, k$  be positive integers, and suppose  $k \leq \log^c x$ . Given any positive constant  $U_1$  there exists a positive constant  $C_1 = C_1(h, c, U_1)$  such that*

$$\sum_{d < x^{1/2}/k \log^{C_1} x} \mu^2(d) h^{\nu(d)} E(x, [k, d]) = O_{h, c, U_1} \left( \frac{x}{\varphi(k) \log^{U_1} x} \right),$$

where  $\mu(d)$  is the Möbius function,  $\varphi(d)$  is the Euler function and  $\nu(d)$  counts the number of prime factor of  $d$ .

*Proof.* The proof is the argument of Lemma 3.5 of [H-R, page 115], replacing  $E(x, kd)$  by  $E(x, [k, d])$  and noting that since  $[k, d] \leq kd$ , we have

$$\sum_{d < x^{1/2}/k \log^{C_1} x} E(x, [k, d]) \leq \sum_{d < x^{1/2}/\log^{C_1} x} E(x, d).$$

We now introduce some notation. Let  $f(x)$ ,  $k$  and  $l$  be as in (2.2). For a squarefree integer  $d$ , let  $D = (d, k)$ , and

$$\rho_{k, l}(d) = \begin{cases} |\{1 \leq m \leq d/D, (m, d/D) = 1, f(m) \equiv 0 \pmod{d/D}\}| & \text{if } D | f(l), \\ 0 & \text{if } D \nmid f(l). \end{cases}$$

It is easy to check that, as in Example 6 in [H-R],  $\rho_{k, l}(d)$  is a multiplicative function.

**Lemma 2.2.** *Let  $l$  and  $k$  be relatively prime integers. Let  $f(x)$  be an irreducible polynomial of degree  $g$  with integer coefficients such that  $(f(l), k) = 1$ . Consider the set  $A = \{f(p) : p \leq x \text{ prime}, p \equiv l \pmod{k}\}$ . Suppose that the function given by  $\rho(d) = |\{1 \leq m \leq d, f(m) \equiv 0 \pmod{d}\}|$  satisfies*

$$(2.3) \quad \rho(p) \leq p - 1 \quad \text{and} \quad \rho(p) < p - 1 \quad \text{if} \quad p \leq g + 1, p \nmid f(l).$$

Then for any squarefree  $d$ , we have the following relations

a) For  $X = \frac{\tilde{l}(x)}{\varphi(k)}$ , we have

$$|A_d| = X \frac{\omega(d)}{d} + R_d,$$

where  $\omega(d) = \rho_{k,l}(d)\varphi(D)\frac{d}{\varphi(d)}$  is multiplicative and

$$|R_d| \leq \rho(d) (E(x, [k, d]) + 1).$$

b) The functions  $\omega(d)$  and  $R_d$  satisfy conditions  $(\Omega_1)$ ,  $(\Omega_2^*(1))$ ,  $(\Omega_3)$  and  $(R(1, \alpha))$ .

*Proof.* a) The proof is the argument in examples 5 and 6 in [H-R]. Note that  $\omega(\cdot)$  is multiplicative since  $\rho_{k,l}(\cdot)$  is multiplicative as remarked above.

b)

• We first verify condition  $(\Omega_1)$ . If  $p|k$ , then  $p \nmid f(l)$  and  $\omega(p) = 0$ . Otherwise,

$$\omega(p) = \begin{cases} \frac{\rho(p)-1}{p-1}p & \text{if } p|f(l) \\ \frac{\rho(p)}{p-1}p & \text{if } p \nmid f(l). \end{cases}$$

By (2.3) we have  $\omega(p) \leq (1 - 1/g)p$  whenever  $p \leq g + 1$ . Meanwhile if  $p \geq g + 2$ , then by Lagrange's Theorem and (2.3) we have  $\rho(p) \leq g$  and so  $\omega(p) \leq \frac{g}{p-1}p \leq (1 - 1/(g + 1))p$ , and so  $\Omega_1$  is satisfied with  $C_1 = g + 1$ .

• Condition  $(\Omega_2^*(1))$  is a trivial consequence of Nagel's result [N], (see [H-R, page 18])  $\sum_{p < x} \rho(p) \log p/p = \log x + O(1)$ , and partial summation.

• Now we explain how to guarantee condition  $(\Omega_3)$ . If  $\mathcal{D}$  is the discriminant of  $f$ , then it is well known that  $\rho(p^2) \leq g\mathcal{D}^2$  (see [H-W]). Hence,

$$|A_{p^2}| \leq |\{n \leq x : f(n) \equiv 0 \pmod{p^2}\}| \ll \frac{x}{p^2} + 1 \ll \frac{X \log X}{p^2} + 1,$$

which trivially gives  $(\Omega_3)$ .

• Now we verify condition  $(R(1, \alpha))$ . We have, by Lagrange's Theorem that  $\rho(p) \leq g$ , and so  $\rho(d) \leq g^{\nu(d)}$ . Therefore, we have by a)

$$(2.4) \quad |R_d| < g^{\nu(d)} (E(x, [k, d]) + 1).$$

**Lemma 2.3.** *Under the hypotheses of Lemma 2.2, we have for  $x \gg 1$*

$$\begin{aligned} & |\{p \leq x \text{ prime} : p \equiv l \pmod{k} : f(p) = P_{2g+1}\}| \\ & \geq \frac{2}{7} \prod_{p|k} \frac{p}{p-1} \prod_{p|f(l)} \frac{1 - (\rho(p) - 1)/(p-1)}{1 - 1/p} \prod_{p|f(l), p \nmid k} \frac{1 - \rho(p)/(p-1)}{1 - 1/p} \frac{x}{\log^2 x}. \end{aligned}$$

*Proof.* Apply Theorem T1 with  $\alpha = 1/2$  and  $r = 2g + 1$ . The result follows then from

$$\frac{\text{li}(x)}{\varphi(k) \log(\text{li}(x)/\varphi(k))} \geq \frac{\text{li}(x)}{\varphi(k) \log \text{li}(x)} \geq \frac{x}{\varphi(k) \log^2 x}.$$

### Proof of Theorem 0.6

Let  $G(x)$  be as in (2.1). In this case we have  $G(-31) = 149$ , and so (2.3) is trivial for this polynomial. Hence we can apply Lemma 2.3 to the polynomial  $G(x)$ ,  $k = 28$ ,  $l = -31$  and  $x = X^{1/3}$  to deduce Theorem 0.6.

### 3. The case $D = 15$ .

Let  $K = \mathbb{Q}(\sqrt{-D})$  be an imaginary quadratic field with discriminant  $-D$ , and let  $H$  be the Hilbert class field of  $K$ . Gross showed, using the theory of complex multiplication ([G1, Theorem 9.1]), that producing an elliptic curve over  $H$  with CM by  $\vee_K$  is the same as giving its  $j$ -invariant together with an algebraic Hecke character of  $H$  with values in  $K$  such that

$$(3.1) \quad \chi(\alpha \vee_H) = N_{H/K} \alpha$$

for all  $\alpha \equiv 1 \pmod{* \mathfrak{M}}$ , where  $\mathfrak{M}$  is some integral ideal of  $H$ . The relation between  $\chi$  and the elliptic curve  $A$  is that

$$\chi(\mathfrak{A}) + \overline{\chi(\mathfrak{A})} = \#k + 1 - \#A(k)$$

for every integral ideal  $\mathfrak{A}$  of  $H$  prime to  $\mathfrak{M}$  (conductor of  $A$ ), where  $k$  is the residue field at  $\mathfrak{A}$ .

When  $D \equiv 3 \pmod{4}$ , let  $\epsilon$  be the quadratic character  $\epsilon : (\vee_K/\sqrt{-D})^* \cong (\mathbb{Z}/D)^* \rightarrow \{\pm 1\}$ , where the last map is the Dirichlet character  $(\frac{-D}{\cdot})$ . So

$$(3.2) \quad \epsilon_D(\alpha \vee_K) = \epsilon(\alpha) \alpha$$

is a well-defined homomorphism from the group of principal ideals of  $K$  to  $K^*$ . Since the norm of ideals of  $H$  to  $K$  are always principal by class field theory, this gives rise a unique Hecke character  $\chi_H = \epsilon_D \circ N_{H/K}$  of  $H$  satisfying the condition (3.1). So there is a unique elliptic curve  $A(D)$  over  $H$  with associated

Hecke character  $\chi_H$  and  $j$ -invariant  $j(A(D)) = j(\frac{1+\sqrt{-D}}{2})$ . Furthermore, [G1, Theorem 10.2] asserts that  $A(D)$  descends to two isogenous elliptic curves over  $F = \mathbb{Q}(j)$ , which we still denote by  $A(D)$ . One can distinguish the two elliptic curves by their minimal discriminants as done by Gross in the case where  $D$  is a prime. On the other hand,  $\epsilon_D$  extends to  $h_D$  so-called canonical Hecke characters of  $K$ , denoted by  $\chi_D$ . Here  $h_D$  is the ideal class number of  $K$ . Canonical Hecke characters differ from each other by ideal class characters. By the theory of complex multiplication, one sees that

$$L(s, A(D)/F) = L(s, \chi_H) = \prod L(s, \chi_D), \quad L(s, A(D)^d/F) = \prod L(s, \chi_{D,d}),$$

where the product runs over all canonical Hecke characters of  $K$ , and  $\chi_{D,d} = \chi_D(\frac{d}{\cdot}) \circ N_{K/\mathbb{Q}}$  is the quadratic twist of  $\chi_D$ . We remark that all sides in the above identities are independent of the choices of  $A(D)$  or  $\chi_D$ .

The arithmetic and the L-functions of  $A(D)$  and its quadratic twists have been extensively studied by Gross, Rorhlich, Rodriguez-Villegas, and the second author among others. For example, it is now known ([G1], [M-R], [M-Y]) that  $A(D)(F)$  has the Mordell-Weil rank 0 or  $h_D$  depending on whether  $D \equiv 7 \pmod{8}$  or  $3 \pmod{8}$ , and that the Tate-Shafarevich group is finite. When  $D$  is a prime number, Gross also determines its torsion group ([G1]) and its minimal model ([G2]). It seems to be of independent interest to extend Gross's work to general  $D$ . In this section we deal with the special case  $D = 15$ , and answer questions 0.3 and 0.4 affirmatively in this case. From now on, let  $K = \mathbb{Q}(\sqrt{-15})$ , and let

$$(3.3) \quad j = j\left(\frac{1 + \sqrt{-15}}{2}\right) = -\frac{191025 + 85995\sqrt{5}}{2}.$$

So  $F = \mathbb{Q}(j) = \mathbb{Q}(\sqrt{5})$ , and  $H = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$  is the Hilbert class field of  $K$ . Let  $\epsilon = \frac{1+\sqrt{5}}{2}$  be a fundamental unit of  $F$ . Then there are algebraic integers  $m, n \in \mathbb{V}_F$  such that ([B, pp 57])

$$m^3 = -\epsilon j \quad \text{and} \quad -3n^2 = j - 1728.$$

Similarly to [G1, page 80], we set for any nonzero number  $c \in F^*$

$$(3.4) \quad E_c : y^2 = x^3 - 9\epsilon mc^2 x + 18\epsilon^2 nc^3.$$

Then  $j(E_c) = j$  and  $\Delta(E_c) = -2^{12}3^9\epsilon^4 c^6$ . Let

$$(3.5) \quad E = E_{\frac{1}{12\epsilon}} : y^2 = x^3 - \frac{1}{16}(15 + 12\sqrt{5})x + \frac{7}{64}(6 + 4\sqrt{5}).$$

Then  $\Delta(E) = -3^3\epsilon^{-2}$ , and  $E_c$  is just the quadratic twist  $E^{12\epsilon c}$  of  $E$ . We mention in passing that the denominators in equation (3.5) are for the purpose of getting the minimal discriminant  $3^3\mathbb{V}_F$  (see Proposition 3.1 below) and can be easily cleared. Indeed,  $E$  is isomorphic to

$$(3.5') \quad E = E^4 : y^2 = x^3 - (15 + 12\sqrt{5})x + 7(6 + 4\sqrt{5}),$$

which has integral coefficients.

**Proposition 3.1.** (1) *The CM elliptic curve  $E$  has minimal discriminant  $3^3\sqrt{F}$  and conductor  $3^2\sqrt{F}$ . In particular, it has good reduction everywhere outside  $3\sqrt{F}$ .*

(2)  *$E$  is  $F$ -isogenous to its Galois conjugate,*

$$E' : y^2 = x^3 - \frac{1}{16}(15 - 12\sqrt{5})x + \frac{7}{64}(6 - 4\sqrt{5}).$$

(3) *The elliptic curve  $E$  is  $F$ -isogenous to the quadratic twist  $E^{-3}$ . In particular,  $E^{-3}$  has minimal discriminant  $3^9\sqrt{F}$  and has good reduction everywhere outside 3.*

*Proof.* Direct calculation gives  $j(E_c) = j$  and  $\Delta(E_c) = -2^{12}3^9\epsilon^4c^6$ . In particular,  $\Delta(E) = -3^3\epsilon^{-2}$ , and so  $E$  has good reduction everywhere outside  $6\sqrt{F}$ . The substitution

$$x = x_1 - \frac{1}{4}, y = y_1 + \frac{1}{2}x_1 + \frac{\epsilon}{2}$$

gives an integral model of  $E$  with the same  $\Delta$ :

$$E : y_1^2 + x_1y_1 + \epsilon y_1 = x_1^3 - x_1^2 - 2\epsilon x_1 + \epsilon.$$

This implies that  $E$  has good reduction at 2. So  $E$  has good reduction everywhere outside  $3\sqrt{F}$ . Notice that  $E$  has CM and thus its conductor is a square, which divides  $\Delta$ . So its conductor is  $3^2\sqrt{F}$ . This proves (1).

To prove (2), we compute the 5th division polynomial of  $E$  using the equation (3.5') and MAGMA. It has a quadratic factor  $x^2 - 2\sqrt{5}x + \frac{6\sqrt{5}}{5} - 1$ . Using the algorithm in [C, page 99], one then finds that  $E$  is 5-isogenous to the elliptic curve

$$y^2 = x^3 - 5^2(15 - 12\sqrt{5})x + 5^37(6 - 4\sqrt{5})$$

over  $F$ , which is isomorphic to  $E'$ . The same procedure shows that  $E'$  is 3-isogenous to  $E^{-3}$ . This, combined with (2), shows that  $E$  is 15-isogenous to  $E^{-3}$ , proving (3). Incidentally, this isogeny becomes the complex multiplication by  $\sqrt{-15}$  over  $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$ .

**Theorem 3.2.**

(1) *The two elliptic curves  $A(15)$  over  $F$  are  $A(15)_1 = E^{-\sqrt{5}(2+\sqrt{5})}$  and  $A(15)_2 = A(15)_1^{-3}$ .*

(2) *There are infinitely many square-free integers  $d$  such that  $A(15)^d(F)$  is infinite and such that the functional equation of  $L(s, \chi_{15}^d)$  has positive sign. Here  $A(15) = A(15)_i$  with  $i = 1, 2$ .*

It is interesting to note that the functional equation for  $L(s, A(15)^d)$  has positive sign. However, it is trivially zero at  $s = 1$  if the root number of  $\chi_{15}^d$

is  $-1$ . It is also interesting to note that  $A(15)$  has a quadratic twist which has good reduction everywhere outside  $3$ , including the prime  $\sqrt{5}$  of  $\mathbb{Q}(\sqrt{5})$ . This can not happen if  $D$  is prime to  $6$ .

*Proof.* Let  $E1 = E^{-\sqrt{5}(2+\sqrt{5})}$ . Since  $-\sqrt{5}(2+\sqrt{5}) \equiv 1 \pmod{4}$ , the quadratic twist does not induce bad reduction at  $2$ , and so the conductor of  $E1$  is  $(3\sqrt{5}\mathcal{V}_F)^2$ . The same proof as in Proposition 3.1 shows that  $E1$  is a CM  $\mathbb{Q}$ -curve, and thus that the scalar restriction  $B = \text{Res}_{F/\mathbb{Q}} E1$  is a CM abelian variety over  $\mathbb{Q}$  with CM by  $H$  such that all its complex multiplications are defined over  $K$ . This implies, by the theory of complex multiplication, that there is an algebraic Hecke character  $\chi$  of  $K$  with values in  $H$  such that

$$L(s, E1) = L(s, B) = L(s, \chi)L(s, \chi^\sigma)$$

where  $\sigma \in \text{Gal}(H/K)$  is nontrivial. Looking at the functional equation of both sides, one sees that the conductor of  $\chi$  is  $\sqrt{-15}\mathcal{V}_K$ , the same as that of  $\chi_{15}$ . So  $\phi = \chi\chi_{15}^{-1}$  is a Hecke character of  $K$  of finite order and with good reduction everywhere. This means that  $\phi$  is an ideal class character of  $K$ . Replacing  $\chi_{15}$  by  $\chi_{15}\phi$  if necessary, we obtain  $\chi = \chi_{15}$ . Recall that  $j(E1) = j$ , so  $E1$  is one of  $A(15)$  over  $F$ , the other one is  $E1^{-3}$ . This proves (1).

To prove (2), we take  $A(15)$  to be  $E1$ . A substitution  $x \mapsto x + \frac{1}{4}(35 + 16\sqrt{5})$  gives

$$E1 : y^2 = \frac{x}{4}f(x),$$

where

$$f(x) = 4x^2 + 105x + 1410 + (48x + 630)\sqrt{5}.$$

Assume that  $x$  is a rational number and that

$$(3.6) \quad f(x) = d_1(a + \sqrt{5})^2$$

with  $a, d_1 \in \mathbb{Q}$ . Then  $(x, \frac{1}{2}(a + \sqrt{5}))$  is a nontrivial  $F$ -rational point of the quadratic twist  $E1^d$  with  $d = xd_1$ . Since  $a, d_1$ , and  $x$  are rational numbers, (3.6) implies

$$\frac{4x^2 + 105x + 1410}{a^2 + 5} = \frac{48x + 630}{2a} = d_1.$$

Substituting  $a = \frac{8x+105}{z}$ , one has then

$$(3.7) \quad A : -15z^2 + (1410 + 105x + 4x^2)z - 192x^2 - 5040x - 33075 = 0$$

and the following fact: If  $(x, z)$  is a rational solution of (3.7) then  $E1^d$  with

$$(3.8) \quad d = 3xz$$

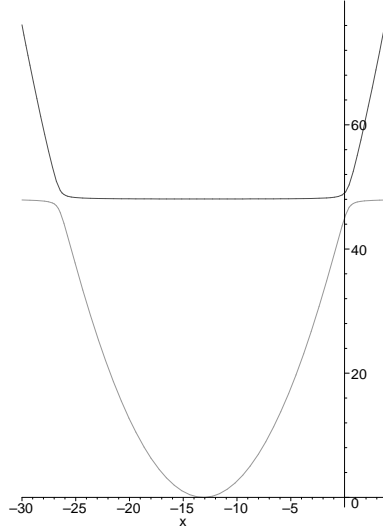
has an  $F$ -rational point which is of infinite order for all but finitely many  $d$  ([Go, Proposition 1]). Now part (2) of the theorem follows from the following steps.

1. The equation (3.7) defines an elliptic curve  $A$  over  $\mathbb{Q}$ . It has two ‘infinite’ points, the horizontal one  $O = [0, 1, 0]$  and the vertical one  $[1, 0, 0]$  in terms of homogeneous coordinates  $[x, z, y]$ . We choose  $O$  to be the identity. One can check that  $Q_0 = (17/28, 2307/49)$  is of infinite order. The graph in the next page is the real locus of  $A$  in the  $(x, z)$ -plane.

2. For each rational point  $P = (x, z) \in A(\mathbb{Q})$ , let  $d(P)$  be the squarefree part of  $3x(P)z(P)$ . Then for every square free integer  $d \neq 0$ , there are only finitely many  $P \in A(\mathbb{Q})$  such that  $d = d(P)$ . Indeed, for a fixed  $d$ , if  $3xz = dy^2$ , then (3.7) gives rise to

$$C : -15d^2y^4 + 3d(1410 + 105x + 4x^2)xy^2 - 27(8x + 105)^2x^2 = 0.$$

This defines an algebraic curve  $C$ , which has 2 double points  $(0, 0)$  and  $(-\frac{105}{8}, 0)$  and is nonsingular everywhere. So the normalization of  $C$  has genus  $6-2 = 4 > 1$  generically and has thus finitely many rational points. This implies that  $A$  produces infinitely many square-free  $d(P)$ s.



3. It is known that the root number of  $\chi_{15}^d$  is the sign of  $d$  when  $(15, d) = 1$ . Since 5 is a square in  $F$ , we can replace  $d$  by  $d/5$  without affecting the curve or the root number. So we only need to make sure that  $3 \nmid d$ . Let  $Q_0^0 = Q_0 = (\frac{17}{28}, \frac{2307}{49}) \in A(\mathbb{Q})$ , and for each integer  $r > 0$ , let  $Q_r^j = (-1)^j 2Q_{r-1}^0$  for  $j = 0, 1$ . For example,

$$Q_1^0 = \left( -\frac{671}{112}, \frac{867}{64} \right), \quad Q_1^1 = \left( -\frac{2269}{112}, \frac{867}{64} \right),$$

and

$$Q_2^0 = \left( \frac{-8520616668059}{290795014496}, \frac{126353913920688}{2639880802441} \right),$$

$$Q_2^1 = \left( \frac{887247537539}{290795014496}, \frac{126353913920688}{2639880802441} \right).$$

**Claim** Let  $x_r^j = x(Q_r^j)$ , and  $z_r = z(Q_r^j)$ . Then the  $x$ -coordinates  $x_r^j$  are relatively prime to 3 (i.e., in  $\mathbb{Z}_3^*$ ), and

$$(3.9) \quad z_r \equiv -6 \pmod{9}, \quad \text{but} \quad z_r \not\equiv -6 \pmod{27}.$$

First notice that by rewriting (3.7) as a polynomial equation of  $x$ , one sees

$$(3.10) \quad x_r^0 x_r^1 = -\frac{15}{4} \frac{(z_r - 45)(z_r - 49)}{z_r - 48}, \quad x_r^0 + x_r^1 = -\frac{105}{4}.$$

So (3.9) would imply that  $x_r^0 x_r^1$  is prime to 3. This implies in turn by (3.10) that  $x_r^j$  is prime to 3. We prove (3.9) by induction. Direct computation using MAPLE gives

$$z(2Q) = \frac{3}{4} \frac{f(x, z)}{g(x, z)},$$

where  $x = x(Q)$ ,  $z = z(Q)$ , and

$$\begin{aligned} f(x, z) = & 64x^6z - 2048x^6 - 80640x^5 + 2800x^5z + 6200zx^4 + 208080x^4 \\ & + 80z^2x^4 + 47565000x^3 + 12600z^2x^3 - 1604400zx^3 - 2400z^3x^2 + 876121425x^2 \\ & - 42374700zx^2 + 617700z^2x^2 - 420336000zx + 6604510500x + 8914500z^2x \\ & - 63000z^3x + 4500z^4 + 59625000z^2 + 21918802500 - 846000z^3 - 1867122000z, \end{aligned}$$

and

$$g(x, z) = x^2(-30z + 1410 + 105x + 4x^2)^2.$$

When  $3 \nmid x$  and  $z$  satisfies (3.9), one sees immediately from the formulas that  $\text{ord}_3 z(2Q) = 1$ . By (3.9), one has  $z \equiv 3 \pmod{9}$  and so

$$z(2Q)/3 + 2 \equiv \frac{3}{x^2 + 3x + 3} \pmod{9}.$$

This proves that  $z(2Q)$  also satisfies (3.9) and thus the claim.

So we see that  $d(Q_r^j)$  is always relative prime to 3. To show that there are infinitely many  $d(Q_r^j) > 0$ , it is enough to make the following observation: If  $x(Q)$  and  $x(-Q)$  are both negative, then at least one of the four numbers  $x(\pm 2Q)$  and  $x(\pm 4Q)$  is positive. This can be easily seen from MAPLE. To be brief, we can assume that  $-105/8 < x(Q) < 0$ . If  $Q$  is in the upper branch, then one of the two numbers  $x(\pm 2Q) > 0$ . Let  $x_0 = -0.768\dots$  be the reflective point of the lower branch of  $A(\mathbb{R})$  on the right of  $x = -105/8$ . When  $-105/8 < x(Q) < x_0$ , the tangent line at  $Q$  is below the curve (concave-up) and hits the curve at  $-2Q$  with  $x(-2Q) > 0$ . When  $x_0 < x < 0$ , the same consideration gives  $x(2Q) < x_0$ . So either  $x(4Q)$  or  $x(-4Q)$  is positive. Since  $x_0$  is not a rational number, we don't need to worry about  $x = x_0$ . This completes the proof of Theorem 3.2.

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