# A SHORT INTRODUCTION TO HILBERT MODULAR SURFACES AND HIRZEBRUCH-ZAGIER DIVISORS

STEPHAN EHLEN

## 1. Modular curves and Heegner Points

The modular curve $Y(1) = \Gamma\backslash\mathbb{H}$ with $\Gamma = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ classifies the equivalence classes of elliptic curves over $\mathbb{C}$. We have an isomorphism:

$$\frac{\mathbb{H}}{\Gamma} \cong \frac{\{\text{elliptic curves over } \mathbb{C}\}}{\{\mathbb{C}\text{-isomorphism}\}} \tag{1}$$

The reason is: every elliptic curve $E/\mathbb{C}$ is corresponds to $\mathbb{C}/L$ with $L \subset \mathbb{C}$ a lattice in $\mathbb{C}$. The points of the elliptic curve are parametrized by the Weierstrass-$\wp$ function which is invariant under translation by $L$ ($\wp$ is an elliptic function). Isomorphic elliptic curves correspond to homothetic lattices and since every lattice is homothetic to a lattice of the form $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{H}$ every equivalence class elliptic curve corresponds to a set of homothetic lattices. Two lattices $L = \mathbb{Z} + \mathbb{Z}\tau$ and $L' = \mathbb{Z} + \mathbb{Z}\tau'$ are homothetic if and only if a matrix $A \in \Gamma$ exists such that

$$\tau' = A\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \tag{2}$$

Thus, dividing through $\Gamma$ gives us exactly the equivalence classes of lattices.

The modular curve $Y(1)$ is non-compact but it can be compactified by 'adding' one point at infinity. The compactified curve is noted by $X(1)$ and it is isomorphic to the Riemann Sphere $\mathbb{P}^1(\mathbb{C})$ (via the $j$-function).

Now we want to study some special points on $Y = Y(1) \subset X = X(1)$ as an introduction:

**Definition 1.1.** Let $D < 0$ be a fundamental discriminant. A point $\tau \in \mathbb{H}$ is called a *Heegner Point* of discriminant $D$ if there exist $A, B, C \in \mathbb{Z}$ with $B^2 - 4AC = D$ such that

$$A\tau^2 + B\tau + C = 0 \tag{3}$$

The images of $\tau$ in $X(1)$ and in $Y(1)$ are also called a Heegner points.

**Proposition 1.2.** *Let $[\tau] \in Y$. Then: $[\tau]$ is a Heegner point if and only if the corresponding modular curve $\mathbb{C}/L$ (unique up to isomorphism) with $L = \mathbb{Z} + \mathbb{Z}\tau$ (unique up to homothety) has complex multiplication by the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$.*

Another correspondence exists between Heegner points of a given discriminant and the ideal class group of a imaginary quadratic field of the same discriminant. We associate to $\tau$ which satisfies definition 1.1 the ideal class of the fractional ideal generated by $A$ and $(B + \sqrt{D})/2$ in $K$ denoted by

$$\left[A, \frac{B + \sqrt{D}}{2}\right] \tag{4}$$

*Date*: September, 25th 2007.

**Proposition 1.3.** *The map*

$$\tau \mapsto \left[A, \frac{B+\sqrt{D}}{2}\right] \tag{5}$$

*is a bijection between Heegner points* $[\tau] \in Y$ *of discriminant* $D$ *and classes of fractional ideals in* $K_D = \mathbb{Q}(\sqrt{D})$ *where* $D < 0$ *is a fundamental discriminant. The inverse map is*

$$\left[A, \frac{B+\sqrt{D}}{2}\right] \mapsto \tau = \frac{-B+\sqrt{D}}{2A} \tag{6}$$

*Particularly:*

$$\#\{[\tau] \in Y \text{ Heegner point }\} = \#\{[\mathfrak{a}] \in \mathrm{Cl}(K_D)\} \tag{7}$$

A generalization of this model are the curves $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ and their compactification $X_0(N)$. They classify pairs of elliptic curves $(E, E', \phi)$ together with a isogeny of degree $N$. Here, $\Gamma_0(N)$ is the following subgroup of $\Gamma(1)$:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \quad (N) \right\} \tag{8}$$

In this case the Heegner points correspond to points $(E, E', \phi) \in Y_0(N) \subset X_0(N)$ where $E$ and $E'$ have the same ring of complex multiplication $\mathrm{End}(E) = \mathrm{End}(E') = \mathcal{O} \subset K = \mathbb{Q}(\sqrt{D})$.

## 2. Hilbert modular surfaces

Now let $d > 1$ be a squarefree integer and let $F = \mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$ be the associated real quadratic number field. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of algebraic integers in $F$. Recall that we have:

$$\mathcal{O}_F = \begin{cases} \mathbb{Z} + \sqrt{d}\mathbb{Z} & \text{if } d \equiv 2,3 \quad (4) \\ \mathbb{Z} + \frac{1+\sqrt{d}}{2}\mathbb{Z} & \text{if } d \equiv 1 \quad (4) \end{cases} \tag{9}$$

and for the discriminant $D = \mathrm{disc}(F)$ we have:

$$D = \begin{cases} 4d & \text{if } d \equiv 2,3 \quad (4) \\ d & \text{if } d \equiv 1 \quad (4) \end{cases} \tag{10}$$

We have two different embeddings of $F$ in $\mathbb{R}$ and we write $x \mapsto x'$ for the conjugation, $\mathrm{N}(x) = xx'$ for the norm.

**Definition 2.1.** The group $\Gamma = \Gamma_F = \mathrm{SL}_2(\mathcal{O}_F)$ is called the *Hilbert modular group*. It can be embedded into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ via the two different real embeddings.

For $z = (z_1, z_2) \in \mathbb{H}^2$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we write

$$\mathrm{N}(cz + d) = \mathrm{N}(cz_1 + d)\mathrm{N}(c'z_2 + d') \tag{11}$$

The *Hilbert modular group* $\Gamma$ acts on $\mathbb{H}^2$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \left( \frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right) \tag{12}$$

The quotient

$$Y = Y(\Gamma) = \Gamma\backslash\mathbb{H}^2 \tag{13}$$

is a complex surface, called *Hilbert modular surface*.

The two real embeddings of $F$ into $\mathbb{R}$ also induce an embedding

$$\mathbb{P}^1(F) \to \mathbb{P}^1(\mathbb{R})^2 \tag{14}$$

And since $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{P}^1(\mathbb{R})^2$ by fractional linear transformations so does $\Gamma$ on $\mathbb{P}^1(F)$. The orbits under the action of $\Gamma$ on $\mathbb{P}^1(F)$ are called the *cusps* of $\Gamma$. Let $(\alpha : \beta) \in \mathbb{P}^1(F)$ and we may assume that $\alpha$ and $\beta$ are integral (otherwise multiply both with their least common denominator). To $(\alpha : \beta)$ we associate the ideal class $[\alpha, \beta] \in \mathrm{Cl}(F)$.

**Proposition 2.2.** *The map $(\alpha : \beta) \mapsto [\alpha, \beta]$ is a bijection between the set of cusps of $\Gamma$ and the ideal class group $\mathrm{Cl}(F)$ of $F$. In particular, the number of cusps of $\Gamma$ equals the class number $h_F$ of $F$.*

Note that $Y(\Gamma)$ is non-compact but it can be compactified by adding a finite number of points, the cusps of $\Gamma$. But, analogously to the one-dimensional case the compactification can be realized by setting $(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})$ and then the quotient

$$X(\Gamma) = \Gamma \backslash (\mathbb{H}^2)^* \tag{15}$$

which can be made into a compact Hausdorff space leads to the *Baily-Borel compactification* of $Y(\Gamma)$.

It can be shown that $X(\Gamma)$ together with a sheaf of rings on $X(\Gamma)$ becomes a normal complex space. One difference to the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is, that $X(\Gamma)$ is not regular. But the singularities can be resolved.

*Remark* 1. Analogously to the modular curve we discussed in the first section the Hilbert modular surface $Y(\Gamma)$ has a moduli interpretation. It is the moduli space for isomorphism classes of abelian surfaces over $\mathbb{C}$ with real multiplication.

Connected with Hilbert modular surfaces are *Hilbert modular forms*:

**Definition 2.3.** A meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is called a meromorphic *Hilbert modular form* of weight $k$, if

$$f(\gamma z) = \mathrm{N}(cz + d)^k f(z) \tag{16}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $f$ is holomorphic it is called a holomorphic Hilbert modular form.

*Remark* 2. A holomorphic Hilbert modular form $f$ has a normally convergent Fourier expansion at the cusps. In contrast to the situation with modular forms for $\mathrm{SL}_2(\mathbb{Z})$ the Götzky-Koecher principle guarantees that a holomorphic Hilbert modular form is also holomorphic at the cusps.

The lecture notes [Bru06] by Bruinier provide a very good introduction into Hilbert modular surfaces as well as Hilbert modular forms. For further information the books of van der Geer [vdG80] and Freitag [Fre90] are recommended.

## 3. Hirzebruch-Zagier divisors

**Definition 3.1.** We define a lattice $L$ in the vector space $V$ of all hermitian matrices over $F$:

$$V = \left\{ A \in M_2(F) \mid A^t = A' \right\} \tag{17}$$

$$L = \left\{ \begin{pmatrix} a & \lambda \\ \lambda' & b \end{pmatrix} \in M_2(F) \mid a, b \in \mathbb{Z}, \lambda \in \mathfrak{d}_F^{-1} \right\} \subset V \tag{18}$$

Where $\mathfrak{d}_F = (\sqrt{D})$ is the different ideal in $F$. Recall that for $\lambda \in \mathfrak{d}_F^{-1}$ and $x \in \mathcal{O}_F$ we have:

$$\mathrm{tr}(x\lambda) = x\lambda + x'\lambda' \in \mathbb{Z} \tag{19}$$

We can define an action of $\Gamma_F$ on $V$:

$$\gamma A = {\gamma'}^t A \gamma \tag{20}$$

The lattice $L$ is invariant under this action.

Now we are ready to define the *Hirzebruch-Zagier divisors* analogously to the Heegner Points (in fact our definition will be nothing else then a special case of the definition of a Heegner divisor):

**Definition 3.2.** Let $m$ be a positive integer. We define:

$$T_A = \left\{(z_1, z_2) \in \mathbb{H}^2 \mid (z_2, 1)A\begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0\right\} \tag{21}$$

$$T_m = \bigcup_{\substack{A\in L \\ \det(A)=m/D}} T_A \tag{22}$$

The *Hirzebruch-Zagier divisor* $T_m$ of discriminant $m$ is defined as the image of $T_m$ in $Y(\Gamma_F)$.

For $z \in T_A$ and $\gamma \in \Gamma_F$ we have $\gamma z \in T_{\gamma A}$. We write $T_m$ for both, the divisor in $\mathbb{H}^2$ and its image, the divisor in $Y(\Gamma_F)$.

Here are some first properties:

**Proposition 3.3.** (1) $T_m = \emptyset \Leftrightarrow \chi_D(m) = -1$
(2) *$T_m$ then it has finitely many components*
(3) *If $m$ is not the norm of an ideal in $\mathcal{O}_F$ then $T_m$ compact*
(4) *If $mn$ is not a square, the curves $T_m$ and $T_n$ meet transversally and we have a well defined intersection number $T_m.T_n \in \mathbb{Q}$*

Now we will cite two very interesting results where the Hirzebruch-Zagier divisors play an important role.

**Definition 3.4.** Let $D = p > 2$ be a prime (thus, $D \equiv 1 \pmod 4$). We define the arithmetical function

$$H_p(n) = \sum_{\substack{x^2 \le 4n \\ x^2 \equiv 4n \pmod p}} H((4n - x^2)/p) \tag{23}$$

where $H(k)$ is the class number of positive definite binary quadratic forms of discriminant $-k$. And we define

$$I_p(n) = \sum_{\substack{\lambda\in\mathcal{O}_p \\ \lambda \gg 0 \\ \lambda\lambda'=n}} \min(\lambda, \lambda') \tag{24}$$

Further let

$$c(n) = H_p(n) + I_p(n) \tag{25}$$

It turns out, that $H_p(N)$ is the intersection number $T_1.T_n$ if $n$ is not a square. If we further consider the compactification $T_n^c$ of $T_n$ in $X(\Gamma_F)$ we get a contribution to the intersection number from the cusps, given by $I_p(n)$. This is already a lot of arithmetical information, but Hirzebruch and Zagier also proved, that these intersection numbers are the coefficients of a modular form:

**Theorem 3.5** (Hirzebruch, Zagier, 1976)**.** *The function*

$$\phi_D(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi n z} \tag{26}$$

*is a modular form of weight 2 and Nebentypus $\chi_D$ for $\Gamma_0(D)$.*

For more details, see the paper of Hirzebruch and Zagier [HZ76] or [vdG80].

Another interesting fact, observed by Richard Borcherds, is, that the Hirzebruch-Zagier divisors occour as divisors of certain meromorphic Hilbert modular forms. More precisely, Borcherds constructed a lift from certain weakly holomorphic modular forms in to Hilbert modular forms with the help of a theta lift. The resulting Hilbert modular form has a infinite product expansion and its divisor is given by a linear combination of Hirzebruch-Zagier divisors.

In teh following theorem $W_0^+(p,\chi_p)$ denotes the space of all weakly holomorphic modular forms for the group $\Gamma_0(p)$ with character $\chi_p$. Weakly holomorphic means holomorphic in $\mathbb{H}$ but meromorphic at the cusps. $\Gamma_0(p)$ is defined as in (8).

**Theorem 3.6** (Borcherds)**.** *Let $f = \sum_{n\gg-\infty} c(n)q^n$ be a weakly holorphic modular form in $W_0^+(p,\chi_p)$ and assume that $c(n) \in \mathbb{Z}$ for all $n < 0$. Then there exists a meromorphic Hilbert modular form $\Psi(z,f)$ for $\Gamma_F$ (with some multiplier system of finite order) such that: The divisor $Z(f)$ of $\Psi$ is determined by the principal part of $f$ at the cusp $\infty$. It equals*

$$Z(f) = \sum_{n<0} c(n)T_{-n} \tag{27}$$

The theorem is discussed in detail in [BB03] and in a more introductory style in [Bru04] and [Bru06] and is included in Theorem 13.3 in [Bor98].

Meromorphic Hilbert modular forms which arise ase liftings of weakly holomorphic modular forms the way Borcherds constructed them, are nowadays called *Borcherds products*. Bruinier proved a strong *converse theorem*:

**Theorem 3.7** (Bruinier)**.** *Let $h$ be a meromorphic Hilbert modular form for $\Gamma_F$, whose divisor $\operatorname{div}(F) = \sum_{n<0} c(n)T_{-n}$ is given by Hirzebruch-Zagier divisors.*

*Then there is a eakly holomorphic modular form $f \in W_0$ with principal part $\sum_{n<0} c(n)q^n$, and, up to a constant multiple, $h$ is equal to the Borcherds lift of $f$.*

For more details on this theorem see [Bru04, Theorem 3.1], [Bru02] and [Bru06].

Please let me know, if you find any typos or mistakes.

## References


[BB03] J. H. Bruinier and M. Bundschuh, *On borcherds products associated with lattices of prime discriminant*, Ramanujan Journal **7** (2003), 49–61.

[Bor98] R. E. Borcherds, *Automorphic forms with singularities on grassmannians*, Invent. Math **132** (1998), 491–562.

[Bru02] J. H. Bruinier, *Borcherds products on o(2,l) and chern classes of heegner divisors*, Springer Lecture Notes in Mathematics, vol. 1789, Springer-Verlag, Berlin, 2002.

[Bru04] J. H. Bruinier, *Infinite products in number theory and geometry*, Jahresber. Dtsch. Mat. Ver **106** (2004), no. 4, 151–184, available at `http://www.mi.uni-koeln.de/~bruinier/publications/borcherds.pdf`.

[Bru06] J. H. Bruinier, *Hilbert modular forms and their applications*, Preprint (2006), available at `http://www.mi.uni-koeln.de/~bruinier/hilbert.pdf`.

[Fre90] E. Freitag, *Hilbert modular forms*, Springer-Verlag, 1990.

[HZ76] F. Hirzebruch and D. B. Zagier, *Intersection numbers of curves on hilbert modular surfaces and modular forms of nebentypus*, Invent. math. **36** (1976), 57–113.

[vdG80] G. v. d. Geer, *Hilbert modular surfaces*, 3rd ed., Erg. d. Mathematik und ihrer Grenzgeb., vol. 16, Springer-Verlag, 1980.


*E-mail address*: `mail@stephanehlen.de`