COMMON ZEROS OF THETA FUNCTIONS AND CENTRAL
HECKE L-VALUES OF CM NUMBER FIELDS OF DEGREE 4
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Tonghai Yang

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Abstract. In this note, we apply the method in [RVY] to construct a family of
infinite many theta series over the Hilbert-Blumenthal modular surfaces with a com-
mon zero. We also relate the nonvanishing of the central L-values of certain Hecke
characters of non-biquadratic CM number fields of degree 4 to the nonvanishing of
theta functions at CM points in the the Hilbert-Blumenthal modular surfaces.

0. Introduction. It is well-known that the theta function $\theta(z) = \sum e(x^2z)$ has
no zeros in the upper half plane, where $e(z) = e^{2\pi i z}$. It was proved in [RVY] that,
for integers $d \equiv 1 \mod 4$ and $k \geq 0$, whether the theta functions

$$\theta_{d,k}(z) = (\text{Im}(2z))^{-\frac{k}{2}} \sum_{(x,d) = 1} \frac{d}{x} H_k(x(\text{Im}(2z))) e(x^2z)$$

vanishes at certain Heegner points is related to the vanishness of Central Hecke L-
values ([RVY, Theorem 0.2]). Here $H_k$ is a suitably normalized Hermite polynomial.
Over a real quadratic field $F = \mathbb{Q}(\sqrt{p})$, one can also define a theta function $\theta_{d,k}$ on
the Hilbert upper half plane $H^2$ in the same manner. In fact, one can replace $(\frac{d}{x})$ by
any quadratic Hecke character of $F$. (See (1) and (10) for the exact meaning). They
are nonholomorphic Hilbert modular forms of weight $k + \frac{1}{2}$ for some congruence
group $\Gamma_d$ over $F$. We remark that $\Gamma_d$ is independent of $k$. The first result involves
the zero locus of these theta functions. We prove that those theta functions have
sometimes a common zero when $k$ varies.

Theorem 0. Let $p = 5, 13, \text{ or } 61$. Let $d \equiv 1 \mod 4$ be a rational integer such that
every prime factor of $d$ is split in either $\mathbb{Q}(\sqrt{-7})$ or $\mathbb{Q}(\sqrt{-7p})$.
(a) If $d < 0$, then $\{\theta_{d,k} : k \geq 0, \text{ even}\}$ has a common zero in $\mathbb{Q}(\sqrt{p}, \sqrt{-7}) \cap \mathbb{Q}_2$.
(b) If $d > 0$, then $\{\theta_{d,k} : k \geq 1, \text{ odd}\}$ has a common zero in $\mathbb{Q}(\sqrt{p}, \sqrt{-7}) \cap \mathbb{Q}_2$.
Here we embed $\mathbb{Q}(\sqrt{p}, \sqrt{-7})$ into $\mathbb{C}^2$ via $z \mapsto (\sigma_1(z), \sigma_2(z))$, where $\sigma_j(\sqrt{p}) = (-1)^{j-1} \sqrt{p}$, and $\sigma_j(\sqrt{-7}) = i \sqrt{7}$ for $j = 1, 2$.

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Theorem is also true if we replace $p$ by $17$ and $-7$ by $-11$. The proof of this theorem and its generalization will be given in section 2. In this note, we will also relate the central Hecke L-value to the values of theta functions at CM points and relate the vanishing of the Hecke L-value to the CM zeros of the theta functions (Theorem 2 and 8).

1. Notation and preliminary. Let assume $F = \mathbb{Q}(\sqrt{p})$ is a real quadratic number field with $p \equiv 1 \mod 4$. We view $F$ as a subfield of $\mathbb{R}$ with $\sqrt{p} > 0$, so the other embedding of $F$ into $\mathbb{R}$ is give by $x = a + b\sqrt{p} \mapsto x' = a - b\sqrt{p}$, $a, b \in \mathbb{Q}$. We embed $F$ into $\mathbb{C}^2$ via $x \mapsto (x, x')$. Let $J^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_i > 0\}$ be the Hilbert upper plane. Given a function $\phi$ on $\mathbb{C}$, we define a function on $\mathbb{C}^2$, still denoted by $\phi$, via $\phi(z) = \phi(z_1)\phi(z_2)$. Let $u$ be a fixed unit of $F$ with $u > 0$ and $u' < 0$. Let $\mu$ be an quadratic Hecke character $\mu$ of $F$ with principal conductor $f\mathcal{O}_F$, then $\mu$ induces a character $\mu : (\mathcal{O}_F/f) \longrightarrow \mathbb{C}^1$. We assume $\mu(-1) = 1$. Multiplying by $u$ if necessary, we may assume that $f$ is totally positive. For a rational integer $k \geq 0$, we define a theta function $\theta_{\mu, k}$ on the Hilbert upper plane $J^2$ via

\[
\theta_{\mu, k}(z) = (\text{Im } (2z))^{-\frac{k}{2}} \sum_{x \in \mathcal{O}_F \setminus \{0\}} \mu(x)H_k(x\sqrt{\text{Im } (2z)})e(x^2z).
\]

Here $e(z) = e^{2\pi iz}$, and $H_k$ is the $k$th Hermite polynomial normalized via [RVY, (0.13)]. The theta function $\theta_{\mu, k}$ is a nonholomorphic Hilbert modular form of weight $k + \frac{1}{2}$ for some congruence group $\Gamma_d$ over $F$ and is holomorphic when $k = 0$ or 1. We notice that $\Gamma_d$ is independent of $k$.

Let $D$ be a totally positive integer in $\mathcal{O}_F$, and let $E = F(\sqrt{-D})$ be a totally imaginary extension of $F$. We assume throughout this paper

2. $DO_F$ is prime, and $E/F$ is ramified only at $DO_F$;
3. Every prime of $F$ dividing $2f\mathcal{O}_F$ is split in $E/F$, and
4. $E$ is generic in the sense of Rohrlich ([Roh, page 519]), that is, $\mathcal{O}_E^* = \mathcal{O}_F^*$ and $\text{CL}(F) \hookrightarrow \text{CL}(E)$ naturally.

Let $\Phi = \{\sigma_1, \sigma_2\}$ be the CM type of $E$ given by

\[
\sigma_1(\sqrt{p}) = \sqrt{p}, \quad \sigma_1(\sqrt{-D}) = i\sqrt{D}
\]

and

\[
\sigma_2(\sqrt{p}) = -\sqrt{p}, \quad \sigma_2(\sqrt{-D}) = i\sqrt{D}'.
\]

Here $\sqrt{a}$ stands for the positive square root of $a$ for $a > 0$. We embed $E$ into $\mathbb{C}^2$ via $z \mapsto (\sigma_1(z), \sigma_2(z))$.

Let $\tilde{\mu} = \mu \circ \tilde{N}_{E/F}$, then $\tilde{\mu}|\mathbb{A}_E^* = 1$ and there is a character $\eta$ of $E^1 \setminus E^1_{\mathbb{A}}$ such that $\tilde{\mu} = \tilde{\eta}$, where $\tilde{\eta}(z) = \eta(z/\tilde{z})$. Let $\chi_{\text{can}}$ be a canonical Hecke character of $E$ of infinite type $\Phi$ in the sense of Rohrlich ([Roh, page 518]). Then the character $\chi_{\text{can}}\tilde{\mu}$ is closely related to CM abelian varieties ([Sh]) and can be some simple properties ([RVY, Lemma 2.0]). Let $\chi = \chi_{\text{can}}|\mathbb{A}_E^*$, then $\chi|\mathbb{A}_E^* = \epsilon$ is the quadratic Hecke character of $F$ associated to $E/F$. Denote $\eta_k = \eta\chi^k|_{E_\mathbb{A}}$, so $\tilde{\eta}_k = \tilde{\eta}\chi^{2k}$. Let $\psi = \prod \psi_v$ the ‘canonical’ additive character of $F_\mathbb{A}/F$ given by

\[
\psi_v(x) = \begin{cases} e^{2\pi ix} & \text{if } v \text{ is real}, \\ e^{-2\pi i\lambda(x)} & \text{if } v \text{ is finite}, \end{cases}
\]

where $\lambda : F_{\mathbb{A}} \to \text{tr} \otimes \mathbb{Q}/\mathbb{Z} \to \otimes \mathbb{Q}/\mathbb{Z}$. Let $\psi' = \psi \circ \text{tr}$. 


Lemma 1. Let $\alpha = \frac{4u}{D\sqrt{p}}$ and $\delta = \sqrt{-D}$. Then for any rational integer $k \geq 0$, the datum $(\chi, \eta_k, \alpha, \delta, \psi, \Phi)$ satisfies (0.4), (0.6), and (0.9) in [RVY].

Proof. It is easy to see that
$$\epsilon_v(\alpha) = \begin{cases} \epsilon_{\sqrt{p}}(\sqrt{p}) & \text{if } v = \sqrt{p}\mathcal{O}_F \text{ or } D\mathcal{O}_F, \\ 1 & \text{otherwise}. \end{cases}$$

Here we have abused the notation by using $\sqrt{p}$ for the place of $F$ corresponding to the prime ideal $\sqrt{p}\mathcal{O}_F$. Since $n(\psi_v) = -1$ when $v = \sqrt{p}\mathcal{O}_F$ and $n(\psi_v) = 0$ otherwise, one has, by [RVY, Lemma 2.3],
$$\epsilon\left(\frac{1}{2}, (\chi\tilde{\eta}_k)_v, \frac{1}{2} \psi_{E_v}(\chi\tilde{\eta}_k)_v(\delta)\right) = \begin{cases} ? & \text{if } v = D\mathcal{O}_F, \\ \epsilon_{\sqrt{p}}(\sqrt{p}) & \text{if } v = \sqrt{\mathcal{O}_F}, \\ 1 & \text{otherwise}. \end{cases}$$

But the global root number $\epsilon\left(\frac{1}{2}, \chi\tilde{\eta}_k\right) = \mu(-1)(-1)^{2k} = 1$ by [RVY, Lemma 2.3]. So $? = \epsilon_{\sqrt{p}}(\sqrt{p})$. Therefore
$$\epsilon\left(\frac{1}{2}, (\chi\tilde{\eta}_k)_v, \frac{1}{2} \psi_{E_v}(\chi\tilde{\eta}_k)_v(\delta)\right) = \epsilon_v(\alpha)$$
for every place $v$ of $F$. This verifies Condition (0.6) in [RVY]. The verification of [RVY, (0.4) and (0.9)] is trivial.

Now applying [RVY, Theorem 2.7], one has

Theorem 2. Notation as above. Assume that $F$ has ideal class number 1. Fix a square root $r$ of $-D$ mod $16f^2$. For every ideal class $C \in \text{CL}(E)$, choose a primitive ideal $\mathfrak{A} \in C^{-1}$ relatively prime to $2f$, and write
$$\mathfrak{A}^2 = [a^2, \frac{b + \sqrt{-D}}{2}]$$
with a totally positive, and
$$b \equiv r \mod 8f^2.$$
Then
$$L(k + 1, (\chi \text{can}\tilde{\mu})^{2k+1}) = \kappa \left| \sum_{C \in \text{CL}(E)} \frac{\mu(a)}{(\chi \text{can}\tilde{\mu})^{2k+1}(\mathfrak{A})} \theta_{\mu,k}(\tau_{\mathfrak{A}}) \right|^2.$$

Here
$$\kappa = \frac{\pi^{2k+2}}{(k!)^2 \sqrt{N_{F/Q}D}} \left( \frac{D}{p^f} \right)^{2k+1},$$
and
$$\tau_{\mathfrak{A}} = \frac{u(b + \sqrt{-D})}{8\sqrt{p^f}a^2}.$$

For a CM type $\Phi$ of $E$, we write $k\Phi$ for $\sum_{\sigma \in \Phi} k\sigma$. According to Rohrlich ([Roh2, page 700]), there is a group homomorphism
$$h_{k\Phi} : \text{CL}(E) \longrightarrow \text{CL}(E^{k\Phi}).$$
Here $E^{k\Phi}$ is the number field generated by $z^{k\Phi} = \prod \sigma(z)^k$, $z \in E^*$. Applying [RVY, Theorem 2.8], one has
Corollary 3. Notation and assumption as in Theorem 2. Assume further that $h_{(2k+1)\Phi}$ is injective. Then the following statements are equivalent.

(a) The central $L$-value $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$.
(b) The global theta lifting $\theta_{\alpha,\chi}(\eta_k) = 0$.
(c) For every ideal class $C \in CL(E)$, $I_C(\eta_k) = 0$.
(d) For every ideal class $C \in CL(E)$, $\tau_\Phi$ is a root of the theta function $\theta_{\mu,k}$.

In general, one has

Proposition 4. Notation as in Lemma 1. If $CL(F) \supset \text{ker } h_{(2k+1)\Phi}$, and the central value $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$, then $\tau = \frac{u(b+\sqrt{-D})}{8\sqrt{pf^2}} \in E$ is a root of the Hilbert modular form $\theta_{\mu,k}$. Here $b \in \mathcal{O}_F$ satisfies $b^2 \equiv -D \mod 16f^2$.

Proof. Applying the argument in the proof of [RVY, Theorem 2.8] to [RVY, Theorem 2.6], one has that $2u(b+\sqrt{-D})$ is a root of $\theta_{\mu,k,\mathcal{O}_F}$, where $\theta_{\mu,k,\mathcal{O}_F}$ is given by [RVY, (2.3)]. Simple manipulation gives

$$\theta_{\mu,k,\mathcal{O}_F}(z) = \gamma' N_{F/\mathbb{Q}}(4f)^{-k} \theta_{\mu,k}(\frac{z}{16f^2})$$

where $\gamma' = \prod_{v} \frac{G(\frac{1}{2}, \psi_{\mu,k})}{\sqrt{q_v}} \in \mathbb{C}$. So $\tau$ is a root of $\theta_{\mu,k}$.

2. Common zeros of theta functions. Recall $F = \mathbb{Q}(\sqrt{p})$ with $p \equiv 1 \mod 4$ being a prime. Let $d \equiv 1 \mod 4$ be a rational integer relatively prime to $p$, and let $e^d$ be the Hecke character of $\mathbb{Q}$ associated to the Dirichlet character ($\frac{d}{\cdot}$), and let $\mu = e^d \circ N_{F/\mathbb{Q}}$. Denote $\theta_{d,k} = \theta_{\mu,k}$, i.e.,

$$\theta_{d,k}(z) = (\text{Im } (2z))^{-\frac{3}{2}} \sum_{x \in \mathcal{O}_F} \left( \frac{d}{x, x'} \right) H_k(x \sqrt{\text{Im } (2z)}) e(x^2 z).$$

For a rational prime number $q \equiv 3 \mod 4$, let $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ and $F_2 = \mathbb{Q}(\sqrt{-q})$. For an integer $k \geq 0$, let $N^k : CL(E) \rightarrow CL(F_2)$ be the map induced by $\mathfrak{A} \mapsto (N_{E/F_2}(\mathfrak{A}))^{2k+1}$.

Proposition 5. Let $q \equiv 3 \mod 4$ be a rational prime number such that $(q/p) = -1$ and every prime factor of $2d$ is split in either $\mathbb{Q}(\sqrt{-q})$ or $\mathbb{Q}(\sqrt{-pq})$. Assume further that $q > 3$ and $CL(F) \supset \text{ker } N^k$. Let $b \in \mathcal{O}_F$ be an algebraic integer in $F$ such that $b^2 \equiv -q \mod 16d^2$.

(a) If $d < 0$ and $k$ is even, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a root of $\theta_{d,k}$.
(b) If $d > 0$ and $k$ is odd, then $\tau = \frac{u(b+\sqrt{-q})}{8\sqrt{pd^2}}$ is a root of $\theta_{d,k}$.

Proof. Since $p$ is a prime, $CL(F)$ is an odd group, and so the canonical map $CL(F) \rightarrow CL(E)$ is injective. Since $q > 3$, $\mathcal{O}_E^\circ = \mathcal{O}_{F_2}^\circ$. So $E$ is generic. Let $\chi_{\text{can},2}$ is a canonical Hecke character of $F_2$, and let $\chi_{\text{can}} = \chi_{\text{can},2} \circ N_{E/F_2}$, then $\chi_{\text{can}}$ is a canonical Hecke character of $E$ of infinite type $\Phi = \{\sigma_1, \sigma_2\}$. Moreover, one has

$$L(s, \chi_{\text{can}}^{2k+1}(\frac{d}{\cdot}) \circ N_{E/\mathbb{Q}}) = L(s, \chi_{\text{can},2}^{2k+1}(\frac{d}{\cdot}) \circ N_{F_2/\mathbb{Q}}) L(s, (\chi_{\text{can},2}^{2k+1}(\frac{d}{\cdot}) \circ N_{F_2/\mathbb{Q}})^{\sigma_2}).$$

Under either condition (a) or (b), the global root number of $\chi_{\text{can},2}^{2k+1}(\frac{d}{\cdot}) \circ N_{F_2/\mathbb{Q}}$ is $-1$, and so the central $L$-value $L(k+1, \chi_{\text{can},2}^{2k+1}(\frac{d}{\cdot}) \circ N_{F_2/\mathbb{Q}}) = 0$. This implies $L(k+1, \chi_{\text{can}}^{2k+1}(\frac{d}{\cdot}) \circ N_{E/\mathbb{Q}}) = 0$. Notice that $h_{(2k+1)\Phi} = N^k$ in this case. Now applying Proposition 4, one proves the theorem.
Corollary 6. Notation as in Proposition 5. Assume \( q > 3 \) and that \( E = \mathbb{Q}(\sqrt{p}, \sqrt{-q}) \) has ideal class number 1.

(a) If \( d < 0 \), then \( \tau = \frac{u(b+\sqrt{-q})}{s\sqrt{pd^2}} \) is a common root of \( \theta_{d,k} \) for all even integers \( k \geq 0 \).

(b) If \( d > 0 \), then \( \tau = \frac{u(b+\sqrt{-q})}{s\sqrt{pd^2}} \) is a common root of \( \theta_{d,k} \) for all odd integer \( k > 0 \).

Proof of Theorem 0 According to [Yam, Table], the biquadratic CM fields satisfying all the conditions in Corollary 6 are \( \mathbb{Q}(\sqrt{p}, \sqrt{-7}) \) with \( p = 5, 13, 61 \) and \( \mathbb{Q}(\sqrt{17}, \sqrt{-11}) \). This proves Theorem 0.

3. Nonbiquadratic CM number fields of degree 4. Notation and assumptions as in section 1. We further assume that \( D \) is not a rational integer. Write \( D = x + y\sqrt{p} > 0 \) with \( x, y \in \frac{1}{2}\mathbb{Z} \), \( x \equiv y \mod \mathbb{Z} \), and \( y \neq 0 \). Then \( D' = x - y\sqrt{p} > 0 \) and \( q = DD' \) is a positive rational prime number. When \( q = p \), \( \tilde{E} \) is cyclic over \( \mathbb{Q} \) and is a subfield of \( \mathbb{Q}(\zeta_p) \). In this case, \( E^\Phi = E \). When \( q \neq p \), Let \( K = E(\sqrt{-D'}) \), and identify it as a subfield of \( \mathbb{C} \) via

\[
\sigma_1 : \quad K \longrightarrow \mathbb{C}; \quad \sqrt{p} \mapsto \sqrt{p}, \quad \sqrt{-D} \mapsto i\sqrt{D}, \quad \sqrt{-D'} \mapsto i\sqrt{D'}.
\]

Let

\[
\delta = \sqrt{-D} = i\sqrt{D}, \quad \delta' = \sqrt{-D'} = i\sqrt{D'},
\]

and

\[
\tilde{\delta} = \delta + \delta', \quad \tilde{\delta}' = \delta - \delta' = -q.
\]

Then \( \delta\delta' = -q \) and \( \tilde{\delta}\tilde{\delta}' = -2yp \). Furthermore, let \( E' = F(\delta') \), \( \tilde{E} = \mathbb{Q}(\sqrt{q}) \), \( \tilde{E} = \tilde{F}(\delta) \), and \( \tilde{E}' = \tilde{F}(\delta') \). Then \( E' \), \( \tilde{E} \), and \( \tilde{E}' \) are all subfields of \( K \) and \( K \) is a Dihedral extension of \( \mathbb{Q} \) of degree 8. Write \( \text{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau \rangle \) with

\[
\sigma : \delta \mapsto \delta', \quad \delta' \mapsto -\delta, \quad \tilde{\delta} \mapsto -\tilde{\delta}', \quad \tilde{\delta}' \mapsto \tilde{\delta}, \quad \sqrt{p} \mapsto -\sqrt{p}, \quad \sqrt{q} \mapsto -\sqrt{q},
\]

and

\[
\tau : \delta \mapsto \delta, \quad \delta' \mapsto -\delta', \quad \tilde{\delta} \mapsto \tilde{\delta}', \quad \tilde{\delta}' \mapsto \tilde{\delta}, \quad \sqrt{p} \mapsto \sqrt{p}, \quad \sqrt{q} \mapsto \sqrt{q}.
\]

Notice \( \Phi = \{1, \sigma\} \). Let \( \tilde{\Phi} = \{1, \tau\} \), then \( \tilde{\Phi} \) is a CM type of \( \tilde{E} \). One has \( E^\Phi = \tilde{E} \) and \( \tilde{E}^\Phi = E \). So there is a map for every integer \( n > 0 \)

\[
h_{n,\Phi} \circ h_{n,\Phi} : \quad \text{CL}(E) \longrightarrow \text{CL}(\tilde{E}) \longrightarrow \text{CL}(E).
\]

Proposition 7. Assume that \( F \) has ideal class number 1.

(a) When \( q \neq p \), the map \( h_{n,\Phi} \circ h_{n,\Phi} \) is the \((2n)\)-th power map.

(b) When \( q = p \), \( h_{n,\Phi}^2 \) is the \((2n)\)-th power map.

Proof. Given an ideal \( \mathfrak{A} \) of \( E \), we also use \( \mathfrak{A} \) for its ideal class in \( \text{CL}(E) \). We only prove the case \( n = 1 \), the others are the same. By definition ([Roh2, page 700]), \( h_{\Phi}(\mathfrak{A}) = (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap \mathcal{O}_E \). Since \( \text{Gal}(K/\tilde{E}) = \{1, \tau\sigma^3\} = \{1, \sigma\tau\} \), and \( \tau \in \text{Gal}(K/E) \), one has

\[
h_{\Phi}(\mathfrak{A})\mathcal{O}_K = (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap (\tau\sigma^3(\mathfrak{A})\tau(\mathfrak{A})\mathcal{O}_K)
= (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap (\sigma\tau(\mathfrak{A})\mathfrak{A}\mathcal{O}_K)
= \mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K.
\]
So

\[ h_{\Phi} \circ h_{\Phi}(\mathfrak{A}) = (\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K)^{\Phi} \cap E \]
\[ = \mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K \tau(\mathfrak{A}\sigma(\mathfrak{A})\mathcal{O}_K) \cap E \]
\[ = \mathfrak{A}^2 \sigma(\mathfrak{A})\tau(\mathfrak{A})\mathcal{O}_K \cap E \]
\[ = \mathfrak{A}^2 \mathcal{O}_K \cap E \]
\[ = \mathfrak{A}^2 N_{E'/E}(\sigma(\mathfrak{A})) \]
\[ = \mathfrak{A}^2 \]

since \( F \) has ideal class number 1. This proves (a). Claim (b) can be proved similarly.

Now applying Corollary 3, one has immediately the following

**Theorem 8.** Assume that \( F \) has ideal class number 1, and that \( E = F(\sqrt{-D}) \) satisfies conditions (2) – (4). Assume further that \( D \) is not a rational number and \( (2(2k + 1), h_E) = 1 \), where \( h_E \) is the ideal class number of \( E \). Then the central value

\[ L(k + 1, (\chi\text{can}\tilde{\mu})^{2k+1}) = 0 \]

if and only if for every ideal class \( C \in \text{CL}(E) \) and a (and every) primitive ideal \( \mathfrak{A} \in C^{-1} \) relatively prime to \( 2f \), the CM point \( \tau_{\mathfrak{A}} \) in Theorem 2 is a root of the Hilbert modular form \( \theta_{\mu,k} \).

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**References**


