CONNECTEDNESS PROPERTIES OF DIMENSION LEVEL SETS

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Abstract. We prove that the set of all points of effective Hausdorff dimension 1 in $\mathbb{R}^n$ $(n \geq 2)$ is connected, and simultaneously that the complement of this set is not path-connected when $n = 2$.

1. Introduction and Results

In [2], Lutz and Weihrauch investigate sets in $\mathbb{R}^n$ defined by the effective Hausdorff dimensions of their elements. They show the following:

Theorem 1.1. *In $\mathbb{R}^n$, the set of points of dimension strictly less than 1 is totally disconnected, as is the set of points of dimension strictly greater than $n-1$.*

Theorem 1.2. *In $\mathbb{R}^n$, the set of points of dimension less than or equal to 1 is path-connected, as is the set of points of dimension greater than or equal to $n-1$.*

Restricting these results to the simplest case of $n = 2$ suggests that the points with effective Hausdorff dimension 1 are somehow topologically numerous. We investigate the properties of the dimension one points further, proving the following results:

Theorem 1.3. *In $\mathbb{R}^n$ $(n \geq 2)$, the set of points of dimension exactly 1 is connected.*

Theorem 1.4. *In $\mathbb{R}^2$, the set of points of dimension not 1 is not path-connected.*

In Section 2, we review the appropriate notions. In Section 3, we prove the following result about the abundance of points of dimension 1, from which the above two results follow.

Theorem 1.5. *If $Z \subseteq R^n$ $(n \geq 2)$ is closed, connected, and has the property that for any open set U with $Z \cap U \neq \emptyset$, $\mathrm{ind}(Z \cap U) \geq n-1$, then Z contains a point of effective Hausdorff dimension 1.*

Note that by fixing $r_0, r_1 \in \mathbb{R}$ relatively random, one can define

$$F = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_0 = r_0,\ x_1 = r_1\}.$$

Then F is a closed set of dimension $n-2$ with no point of effective Hausdorff dimension less than 2. So in one sense, Theorem 1.5 is optimal.

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2. Semi-measures, Complexity and Dimension

Throughout the rest of the paper, let n be a fixed positive integer greater than one.

Convention 2.1. *ε denotes the empty string in $2^{<\omega}$.*
λ denotes Lebesgue measure on $\mathbb{R}$.
$\pi_i : \mathbb{R}^n \to \mathbb{R}$ denotes projection onto the ith coordinate.

Definition 2.2. We call a function $\mu : (2^{<\omega})^n \to \mathbb{R}_{\geq 0}$ a *semi-measure* if

$$\mu(\varepsilon, \varepsilon, \ldots, \varepsilon) \leq 1$$
$$\mu(\sigma_0, \ldots, \sigma_i, \ldots, \sigma_{n-1}) \geq \mu(\sigma_0, \ldots, \sigma_i {}^\frown 0, \ldots, \sigma_{n-1}) + \mu(\sigma_0, \ldots, \sigma_i {}^\frown 1, \ldots, \sigma_{n-1}).$$

A semi-measure is *enumerable* if it is computable from below.

A semi-measure is *optimal* if it multiplicatively dominates all enumerable semi-measures.

Henceforth, μ will denote an optimal, enumerable semi-measure.

Definition 2.3. For $(\sigma_0, \ldots, \sigma_{n-1}) \in (2^{<\omega})^n$, define the *$KM$-complexity* as

$$KM(\sigma_0, \ldots, \sigma_{n-1}) := -\log \mu(\sigma_0, \ldots, \sigma_{n-1})$$

Note that KM has the pleasing property that if $\sigma_i \subseteq \tau_i$ for all i, then

$$KM(\sigma_0, \ldots, \sigma_{n-1}) \leq KM(\tau_0, \ldots, \tau_{n-1}).$$

Definition 2.4. For $f = (f_0, \ldots, f_{n-1}) \in (2^\omega)^n$, define the *effective Hausdorff dimension* as

$$\dim_H(f) := \liminf_n \frac{KM(f_1 \upharpoonright n, \ldots, f_{n-1} \upharpoonright n)}{n}.$$

Identifying points in $[0,1)$ with points in 2^ω via binary expansion, we define the effective Hausdorff dimension of points in $[0,1)^n$. It is easily verified that the choice of binary expansion (when more than one exist) has no effect on the dimension. It is also seen that translation by a rational amount in a direction parallel to an axis has no effect on the dimension, so we extend this notion to $\mathbb{R}^n$ via such translations.

Just as we use binary expansion to identify points, we will also identify sets. Given $\sigma \in 2^\omega$, let $[\sigma] = \{f \in 2^\omega : \sigma \prec f\}$. We will identify $[\sigma]$ with the interval of reals whose binary expansions are contained in $[\sigma]$. That is, $[\sigma]$ is identified with $\{0.f \in \mathbb{R} : f \in [\sigma]\}$. Note that $\lambda([\sigma]) = 2^{-|\sigma|}$.

It will be convenient to partition $\mathbb{R}^n$ as:

$$\mathfrak{R}^n_m = \{x \in \mathbb{R}^n : \text{exactly } m\text{-many coordinates of } x \text{ are rational}\}$$

Our definition of effective Hausdorff dimension differs from that used in [2], but the two notions are equivalent. While we constructed dimension on $(2^\omega)^n$ and then identified this space with $\mathbb{R}^n$ in the natural way, Lutz and Weihrauch defined dimension directly upon $\mathbb{R}^n$. They also base their notion of dimension on Kolmogorov complexity, while we use KM-complexity. The reader is referred to [3] for the equivalence of martingale defined dimension and complexity defined dimension, and to [1] for further reading on KM-complexity and its relation to Kolmogorov complexity.

We also make heavy use of (classical) inductive dimension. The necessary background can be obtained from chapter 3 of [4], although we repeat the necessary results here.

For $X \subseteq \mathbb{R}^n$, let $\mathrm{ind}(X) \in \{-1, 0, 1, \ldots, n\}$ denote the inductive dimension of a set X. The definition is such that $\mathrm{ind}(X) = -1$ only when $X = \emptyset$.

Proposition 2.5 ([4, Proposition 3.2.10])**.** $\mathrm{ind}(\mathfrak{R}^n_m) = 0$.

Proposition 2.6 ([4, Corollary 3.1.7])**.** *If* $\mathrm{ind}(X) = n$*, then* X *is not contained in the union of* n*-many sets each of inductive dimension 0.*

Definition 2.7. If Y is connected, say X *separates* Y if $Y - X$ is not connected.

Proposition 2.8 ([4, Theorem 3.7.6])**.** *If* $H \subseteq \mathbb{R}^n$ *is open and connected, and* X *separates* H*, then* $\mathrm{ind}(X) \geq n - 1$.

Proposition 2.9 ([4, Theorem 3.2.5])**.** *If* $X \subseteq \mathbb{R}^n$ *is closed and* $\mathrm{ind}(X) > 0$*, then* X *is not totally disconnected.*

Proposition 2.10 ([4, Theorem 3.2.5])**.** *If* $\mathrm{ind}(X) = 0$*, then* X *is totally disconnected.*

3. Proof of Results

Our main result is Theorem 1.5. The main tools to proving this are the following two lemmas. They both say, in a sense, that even if Z has small intersection with a given region, it will have large intersection with a nearby region.

Definition 3.1. Let $C, D \subset R^n$ be distinct closed n-cubes. Call D *adjacent* to C if D is a translation of C, and there is some point v which is a vertex of both C and D.

Note that any given n-cube has $3^n - 1$ adjacent n-cubes.

Lemma 3.2. *Let* $C \subset R^n$ *be a closed* n*-cube aligned with the axes (i.e.,* C *is a translation of* $[0, a]^n$ *for some* a*). Let* $\{D_j\}_{j<3^n-1}$ *be the collection of adjacent* n*-cubes.*

Let $Z \subseteq R^n$ *be a closed, connected set. If* $Z \cap C \neq \emptyset$*, but* $Z \not\subseteq C \cup \bigcup_j D_j$*, then for some* i *and some* D_j*,*

$$\lambda(\pi_i(Z \cap D_j)) \geq \frac{a}{3^{n-1}}. \tag{†}$$

Proof. Consider $\pi_i(D_j)$. Note that there is some b_i such that

$$\pi_i(D_j) \in \{[b_i, b_i + a], [b_i + a, b_i + 2a], [b_i + 2a, b_i + 3a]\}$$

for all j. Let

$$F_i^0 = \bigcup_{\pi_i(D_j)=[b_i,b_i+a]} D_j,$$

and

$$F_i^1 = \bigcup_{\pi_i(D_j)=[b_i+2a,b_i+3a]} D_j$$

Note that 3^{n-1} many D_j participate in each F_i^*. If $\pi_i(F_i^0 \cap Z) = [b_i, b_i + a]$ or $\pi_i(F_i^1 \cap Z) = [b_i + 2a, b_i + 3a]$, then by additivity of λ, some D_j must satisfy (†).

If instead $\pi_i(F_i^0 \cap Z) \subsetneq [b_i, b_i + a]$ and $\pi_i(F_i^1 \cap Z) \subsetneq [b_i + 2a, b_i + 3a]$, then for some c_i^0, c_i^1,

$$\pi_i^{-1}(c_i^0) \cap F_i^0 \cap Z = \emptyset$$

and

$$\pi_i^{-1}(c_i^1) \cap F_i^1 \cap Z = \emptyset.$$

If these exist for every i, then

$$\bigcup_i (\pi_i^{-1}(c_i^0) \cap F_i^0) \cup (\pi_i^{-1}(c_i^1) \cap F_i^1)$$

separates Z, contradicting connectedness. □

Note: The condition that Z be closed is far more than is necessary, of course. The only place we use it in the above is to imply that $\pi_i(Z)$ is measurable. However, we will only be applying this lemma for closed Z.

Lemma 3.3. *Let $C \subset R^n$ be a closed n-cube aligned with the axes (i.e., C is a translation of $[0, a]^n$ for some a). Let $\{D_j\}_{j<3^n-1}$ be the collection of adjacent n-cubes.*

Let $Z \subseteq R^n$ be closed with the property that for any open set U with $Z \cap U \neq 0$, $\mathrm{ind}(Z \cap U) \geq n - 1$. If $Z \cap C \neq \emptyset$, but $Z \not\subseteq C \cup \bigcup_j D_j$, then for some D_j, $Z \cap D_j$ contains a point of dimension at most 1.

Proof. Let $D = \mathrm{interior}(\bigcup_j D_j)$. By connectedness, Z intersects D. It suffices to show:

$$Z \cap D \cap \mathfrak{R}_n^n \neq \emptyset \text{ or } Z \cap D \cap \mathfrak{R}_{n-1}^n \neq \emptyset.$$

Suppose not. Then $Z \cap D \subseteq \bigcup_{j<n-1} \mathfrak{R}_j^n$. But then by Propositions 2.5 and 2.6, this contradicts the hypothesis on Z. □

We now prove the main result.

Proof of Theorem 1.5. We build $x_0, \ldots, x_{n-1} \in \mathbb{R}$ in stages by building sequences $\{\sigma_i^0\}_{i\in\omega}, \ldots, \{\sigma_i^{n-1}\}_{i\in\omega}$ with each $\sigma_i^m \in 2^{<\omega}$. For a fixed i, all the σ_i^m will have the same length, while for a fixed m, $\lim_i |\sigma_i^m| = \infty$. However, it will not necessarily be the case that $\sigma_i^m \subseteq \sigma_{i+1}^m$. Indeed, $\lim_i \sigma_i^m \upharpoonright s$ may not exist.

So for each σ_i^m, we shall consider a point $y_i^m \in [\sigma_i^m]$ (recalling that $[\sigma_i^m]$ is identified with a closed subset of $\mathbb{R}$) and take $x_m = \lim_i y_i^m$. Because the diameter of the $[\sigma_i^m]$ goes to zero, any choice of y_i^m will have the same limit. Our point of dimension 1 will then be $(x_0, \ldots, x_{n-1})$.

At every stage, our construction employs one of two possible strategies: one strategy is for ensuring that the complexity of $(x_0, \ldots, x_{n-1})$ is not too low, while the other ensures that the complexity is not too high.

Strategy 1 (not too low):

Given $\sigma_i^0, \ldots, \sigma_i^{n-1}$ each of length ℓ with $D = [\sigma_i^0] \times \cdots \times [\sigma_i^{n-1}]$ satisfying (†) for some π, without loss of generality assume it satisfies it for π_0.

Suppose we wish to extend by k-many bits, for some k. We consider all possible extensions of $\sigma_i^0, \ldots, \sigma_i^{n-1}$. Clearly we are not interested in extensions which take us away from Z. So consider

$$E = \{(\tau^0, \ldots, \tau^{n-1}) \in (2^k)^n : [\sigma_i^0 {}^\frown \tau^0] \times \cdots \times [\sigma_i^{n-1} {}^\frown \tau^{n-1}] \cap Z \neq \emptyset\}$$

By assumption, $|E| \geq |\pi_0(E)| \geq 2^k/3^{n-1}$. So there exist some $\tau^0, \ldots, \tau^{n-1}$ such that

$$\frac{2^k}{3^{n-1}}\mu(\sigma_i^0 \frown \tau^0, \ldots, \sigma_i^{n-1} \frown \tau^{n-1}) \leq \mu(\sigma_i^0, \ldots, \sigma_i^{n-1}).$$

Thus

$$KM(\sigma_i^0 \frown \tau^0, \ldots, \sigma_i^{n-1} \frown \tau^{n-1}) \geq KM(\sigma_i^0, \ldots, \sigma_i^{n-1}) + k - (n-1)\log 3.$$

Strategy 2 (not too high):

Given $\sigma_i^0, \ldots, \sigma_i^{n-1}$ each of length ℓ with $Z \cap [\sigma_i^0] \times \cdots \times [\sigma_i^{n-1}]$ containing a point $(d_0, \ldots, d_{n-1})$ of dimension at most 1, note that $\sigma_i^k \prec d_k$.

If $(d_0, \ldots, d_{n-1})$ has dimension exactly 1, the proof is complete. If it has dimension less than one, then there exists some $m \geq i$ such that

$$KM(d_0 \upharpoonright m, \ldots, d_{n-1} \upharpoonright m) \leq m.$$

Assuming i is not such an m, choosing the least such m results in

$$KM(d_0 \upharpoonright m, \ldots, d_{n-1} \upharpoonright m) \geq m - 1,$$

because KM can only increase as m increases.

Construction:

Choose some $\sigma_0^0, \ldots, \sigma_0^{n-1}$ all of the same length such that $D = [\sigma_0^0] \times \cdots \times [\sigma_0^{n-1}]$ satisfies (†) for some π, and such that $Z \not\subseteq D$.

At stage i, if $KM(\sigma_i^0, \ldots, \sigma_i^{n-1}) \leq |\sigma_i^0|$, use Lemma 3.2 to replace $\sigma_i^0, \ldots, \sigma_i^{n-1}$ with adjacent strings satisfying (†) for some π. Then follow strategy 1 to generate $\sigma_{i+1}^0, \ldots, \sigma_{i+1}^{n-1}$ of length $|\sigma_i^0| + i$.

Otherwise, use Lemma 3.3 to replace $\sigma_i^0, \ldots, \sigma_i^{n-1}$ with adjacent strings such that $Z \cap [\sigma_i^0] \times \cdots \times [\sigma_i^{n-1}]$ contains a point of dimension at most 1. Then follow strategy 2, either generating $\sigma_{i+1}^0, \ldots, \sigma_{i+1}^{n-1}$ or finding a point of dimension 1 and ending the construction.

Take $(x_0, \ldots, x_{n-1})$ to be the limit of $(y_i^0, \ldots, y_i^{n-1}) \in [\sigma_i^0] \times \cdots \times [\sigma_i^{n-1}]$ as previously discussed. This is our desired point.

Verification:

Clearly if we halt early via some strategy 2, the construction has succeeded. So henceforth we assume this does not happen.

There are several points to check. First, we must show that the x_k actually exist. This is an unfortunately involved proof for what is actually a fairly simple idea: for $j \geq i$, consider how $\sigma_j^0 \upharpoonright |\sigma_i^0|$ can change through the use of the two lemmas. It can be changed directly at stage $i+1$ (when we trade the cube σ_i^0 is a part of for an adjacent cube), or it can be changed indirectly at stage $j > i$ (when we trade a small cube within σ_i^0 for a small cube outside of σ_i^0). The indirect changes add up in a geometric way, and so they will only occur at one boundary of the cube of $\sigma_{i+1}^0 \upharpoonright |\sigma_i^0|$. So either $\sigma_j^0 \upharpoonright |\sigma_i^0|$ stabilizes, or it switches infinitely between two adjacent cubes which share a boundary. Either way, we see that the limit exists.

Now we make the above argument more rigorous. Without loss of generality, we consider only x_0. For a string $\sigma \in 2^\ell$, let $\mathrm{succ}(\sigma)$ denote the lexicographic successor of σ in 2^ℓ and $\mathrm{pred}(\sigma)$ denote the lexicographic predecessor of σ in 2^ℓ.

Claim. Let $|\sigma_i^0| = \ell$. Then for any $j \geq i$, $\sigma_j^0 \upharpoonright \ell$ is one of σ_i^0, $\mathrm{succ}(\sigma_i^0)$, $\mathrm{succ}(\mathrm{succ}(\sigma_i^0))$, $\mathrm{pred}(\sigma_i^0)$, or $\mathrm{pred}(\mathrm{pred}(\sigma_i^0))$.

Proof. Let $\ell_k = |\sigma_k^0|$. Because of the use of Lemma 3.2 or 3.3 in the construction, $\sigma_{k+1}^0 \upharpoonright \ell_k$ need not be σ_k^0, but if not, the two strings will be adjacent in 2^{ℓ_k}. So

$$\inf[\sigma_{k+1}^0 \upharpoonright \ell_k] = \inf[\sigma_k^0] + a_k 2^{\ell_k},$$

where $a_k \in \{-1, 0, 1\}$.

Since $[\sigma_{k+1}^0]$ has diameter $2^{-\ell_{k+1}}$, we have

$$\inf[\sigma_{k+1}^0 \upharpoonright \ell_k] \leq \inf[\sigma_{k+1}^0] \leq \inf[\sigma_{k+1}^0 \upharpoonright \ell_k] + 2^{-\ell_k} - 2^{-\ell_{k+1}}.$$

Thus,

$$\inf[\sigma_i^0] + \sum_{i \leq k < j} a_k 2^{-\ell_k} \leq \inf[\sigma_j^0] \leq \inf[\sigma_i^0] + 2^{-\ell_i} - 2^{-\ell_j} + \sum_{i \leq k < j} a_k 2^{-\ell_k}.$$

Taking a_k to be worst, we see

$$\inf[\sigma_i^0] - 2 \cdot 2^{-\ell_i} < \inf[\sigma_j^0] < \inf[\sigma_i^0] + 3 \cdot 2^{-\ell_i}.$$

So $\mathrm{pred}(\mathrm{pred}(\sigma_i^0)) \leq \sigma_j^0 \upharpoonright \ell_i \leq \mathrm{succ}(\mathrm{succ}(\sigma_i^0))$. □

Claim. For every i, take $y_i^0 \in [\sigma_i^0]$. Then $x_0 = \lim_i y_i^0$ exists.

Proof. Again, let $\ell_i = |\sigma_i^0|$.

For any $j \geq i$, $\sigma_j^0 \upharpoonright \ell_i$ must be one of the five above values. Then consider the closed interval $J_i = [\mathrm{pred}(\mathrm{pred}(\sigma_i^0)] \cup [\mathrm{pred}(\sigma_i^0)] \cup [\sigma_i^0] \cup [\mathrm{succ}(\sigma_i^0] \cup [\mathrm{succ}(\mathrm{succ}(\sigma_i^0))]$. J_i has diameter $5 * 2^{-\ell_i}$, and for any $j \geq i$, $y_j^0 \in J_i$. Thus $\lim_i y_i^0$ converges. □

Next we must show that our point lies on Z.

Claim. $(x_0, \ldots, x_{n-1}) \in Z$.

Proof. By construction, $\left([\sigma_i^0] \times \cdots \times [\sigma_i^{n-1}]\right) \cap Z \neq 0$ for any i. Thus we can take $(y_i^0, \ldots, y_i^{n-1}) \in Z$. Since Z is closed, $(x_0, \ldots, x_{n-1}) = \lim_i (y_i^0, \ldots, y_i^{n-1}) \in Z$. □

Third, we must show that $\dim_H(x_0, \ldots, x_{n-1}) = 1$.

Claim. $\dim_H(x_0, \ldots, x_{n-1}) \geq 1$.

Proof. Our initial strings $\sigma_0^0, \ldots, \sigma_0^{n-1}$ have some complexity $KM(\sigma_0^0, \ldots, \sigma_0^{n-1}) = A$. When we follow strategy 1 at stage i, the length of our strings increase by i many bits, and the complexity increases by at least $i - (n-1)\log 3$. When we follow strategy 2, our resulting strings have length ℓ, and our resulting complexity is at least $\ell - 1$. Replacing all the σ_i^m with adjacent strings changes the complexity by at most $2 \log |\sigma_i^0|$.

So let $\ell_i = |\sigma_i^0|$ and let i_0 be the last stage before stage i at which strategy 2 was followed. Then

$$\begin{aligned} KM(\sigma_i^0, \ldots, \sigma_i^{n-1}) &\geq (\ell_{i_0} - 1) + (\ell_i - \ell_{i_0}) - (i - i_0)((n-1)\log 3 + 2 \log \ell_i) \\ &\geq \ell_i - i((n-1)\log 3 + 2\log \ell_i). \end{aligned}$$

If there is no such stage i_0, then

$$\begin{aligned} KM(\sigma_i^0, \ldots, \sigma_i^{n-1}) &\geq A + \ell_i - \ell_0 - i((n-1)\log 3 + 2\log \ell_i) \\ &\geq \ell_i - \ell_0 - i((n-1)\log 3 + 2\log \ell_i). \end{aligned}$$

Note that by construction, strategy 2 will never be employed in successive stages. So at stage i, strategy 1 will have been used at least every other stage. Further, since strategy 1 used at stage j always increases the length of the strings by j,

$\ell_i \geq i^2/4$. Thus the $-i((n-1)\log 3 + 2\log \ell_i)$ in the above is a lower order term (recalling that n is constant), and so

$$\liminf_i \frac{KM(\sigma_i^0, \dots, \sigma_i^{n-1})}{\ell_i} \geq 1.$$

Now consider some ℓ_i. Then

$$x_0 \upharpoonright \ell_i \in \{\sigma_i^0, \operatorname{succ}(\sigma_i^0), \operatorname{succ}(\operatorname{succ}(\sigma_i^0)), \operatorname{pred}(\sigma_i^0), \operatorname{pred}(\operatorname{pred}(\sigma_i^0))\},$$

and similarly for $x_1, \dots, x_{n-1}$. So

$$|KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i) - KM(\sigma_i^0, \dots, \sigma_i^{n-1})| \leq 4\log \ell_i.$$

So

$$KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i) \geq \ell_i - \ell_0 - i((n-1)\log 3 - 2\log \ell_i) - 4\log \ell_i,$$

and thus

$$\liminf_i \frac{KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i)}{\ell_i} \geq 1.$$

Finally, consider some k with $\ell_i \leq k < \ell_{i+1}$. If stage i follows strategy 1, then $k - \ell_i < i$, and thus

$$\begin{aligned}\frac{KM(x_0 \upharpoonright k, \dots, x_{n-1} \upharpoonright k)}{k} &\geq \frac{KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i)}{k} \\ &> \frac{KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i)}{\ell_i + i} \\ &\geq \frac{KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i)}{\ell_i + 2\sqrt{\ell_i}}.\end{aligned}$$

If stage i follows strategy 2, then

$$KM(\sigma_{i+1}^0 \upharpoonright k, \dots, \sigma_{i+1}^{n-1} \upharpoonright k) > k,$$

since ℓ_{i+1} will be least such that the above does not hold. Thus

$$\begin{aligned}\frac{KM(x_0 \upharpoonright k, \dots, x_{n-1} \upharpoonright k)}{k} &\geq \frac{KM(\sigma_{i+1}^0 \upharpoonright k, \dots, \sigma_{i+1}^{n-1} \upharpoonright k) - 4\log k}{k} \\ &> \frac{k - 4\log k}{k}.\end{aligned}$$

So

$$\dim_H(x_0, \dots, x_{n-1}) = \liminf_k \frac{KM(x_0 \upharpoonright k, \dots, x_{n-1} \upharpoonright k)}{k} \geq 1. \qquad \square$$

Claim. $\dim_H(x_0, \dots, x_{n-1}) \leq 1$.

Proof. Suppose not. Then for some i_0 and all $i > i_0$,

$$KM(x_0 \upharpoonright \ell_i, \dots, x_{n-1} \upharpoonright \ell_i) > \ell_i + 4\log \ell_i.$$

But in this case, $KM(\sigma_i^0, \dots, \sigma_i^{n-1}) > \ell_i$, and so at stage $i+1$, strategy 2 will be invoked, resulting in $KM(\sigma_{i+1}^0, \dots, \sigma_{i+1}^{n-1}) \leq \ell_{i+1}$, and thus

$$KM(x_0 \upharpoonright \ell_{i+1}, \dots, x_{n-1} \upharpoonright \ell_{i+1}) \leq \ell_{i+1} + 4\log \ell_{i+1},$$

contradicting our above assumption about i_0. $\square$

Thus $\dim_H(x_0, \dots, x_{n-1}) = 1$. This completes the proof. $\square$

Proof of Theorem 1.3. Let $X \subset \mathbb{R}^n$ be the set of points of dimension 1.

Suppose A, B are open sets in $\mathbb{R}^n$ such that $A \cap X$ and $B \cap X$ partition X. Then $X \subseteq A \cup B$ and $A \cap B \cap X = \emptyset$. But X is dense, so $A \cap B = \emptyset$.

Let $Z' = \text{ bd } \overline{A}$. Then Z' separates $\mathbb{R}^n$. Let Z be a non-trivial component of Z' (Propositions 2.8 and 2.9). Then for any open set U such that $U \cap Z \neq \emptyset$, $\overline{A}$ intersects U but is not dense in U. So $Z \cap U$ separates U, and thus $\text{ind}(Z \cap U) \geq n-1$.

By the above theorem, Z contains a point of dimension 1, and since $Z \subseteq \mathbb{R}^n - (A \cup B)$, this contradicts our choice of A and B. □

Proof of Theorem 1.4. Suppose f is any non-constant path in $\mathbb{R}^2$. Its image is a connected, locally connected set. Thus in any neighborhood U with $\text{im} f \cap U \neq \emptyset$, $\text{ind}(\text{im} f \cap U) \geq 1$ (Proposition 2.10), which in this case means at least $n - 1$. So by the theorem, it contains a point of dimension 1. □

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