LIMITWISE MONOTONIC FUNCTIONS, SETS, AND DEGREES ON COMPUTABLE DOMAINS

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Abstract. We extend the notion of limitwise monotonic functions to include arbitrary computable domains. We then study which sets and degrees are *support increasing* (*support strictly increasing*) limitwise monotonic on various computable domains.

As applications, we provide a characterization of the sets S with computable *increasing η-representations* using support increasing limitwise monotonic sets on $\mathbb{Q}$ and note relationships between the class of *order-computable* sets and the class of support increasing (support strictly increasing) limitwise monotonic sets on certain domains.

1. Introduction

Limitwise monotonic functions have become an increasingly prominent tool in the literature, with applications in linear orders (see [1], [5], and [8]), computable model theory (see [2], [6] and [10]), and computable algebra (see [7] and [9]), for example. In all this work, the limit functions have domain ω and the approximation functions have domain $\omega \times \omega$.

In this paper, we make the natural extension of limitwise monotonic functions to arbitrary computable domains. Though no new limitwise monotonic sets are introduced (as any computable linear order has a presentation with universe ω), additional structure is produced when certain natural properties are required of the limit function.

After providing necessary background definitions (Section 2), we review known results and demonstrate a basis result for limitwise monotonic sets (Section 3). We show that new support increasing (support strictly increasing) limitwise monotonic sets are introduced when transitioning from ω^α to $\omega^{\alpha+1}$ (Section 4.1), while no new support increasing (support strictly increasing) limitwise monotonics sets are introduced when transitioning from $\omega^* \cdot \omega$ to $\mathbb{Q}$ (Section 4.2). We then study which sets and degrees are support increasing (support strictly increasing) limitwise monotonic on the rationals (Section 5) and on well-orders (Section 6). Lastly, as applications, we provide a characterization of the sets with computable *increasing η-representations* using support increasing limitwise monotonic sets on $\mathbb{Q}$ (Section 7) and study the relationship between the *order-computable* sets and the support increasing (support strictly increasing) limitwise monotonic sets on various domains (Section 8). We finish with open questions (Section 9).

Key words and phrases. limitwise monotonic function, η-representation.

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2. Background

We begin by introducing some requisite terminology.

Definition 2.1. Let $\mathcal{D} = (D : \prec)$ be a linear order. If $F : D \to \omega$ is any total function, the set $\operatorname{supp}(F) = \{x \in D : F(x) > 0\}$ is the *support* of F.

A total function $F : D \to \omega$ is *support increasing* (*support strictly increasing*) if F satisfies $F(x) \leq F(y)$ $(F(x) < F(y))$ whenever $x \prec y$ and $x, y \in \operatorname{supp}(F)$, the range of F is unbounded, and $\operatorname{supp}(F)$ has order type ω.

Definition 2.2. Let $\mathcal{D} = (D : \prec)$ be a fixed computable presentation of a computable linear order. A function $F : D \to \omega$ is **d**-*limitwise monotonic* if there is a total **d**-computable function $f : D \times \omega \to \omega$ satisfying $f(x, s) \leq f(x, s+1)$ such that $F(x) = \lim_s f(x, s)$ exists for all $x \in D$.

A set $S \subseteq \omega$ is a *(support increasing / support strictly increasing)* **d**-*limitwise monotonic set on* $\mathcal{D}$ if there is a (support increasing / support strictly increasing) **d**-limitwise monotonic function F on $\mathcal{D}$ whose range is $S \cup \{0\}$.

A degree **a** is a *(support increasing / support strictly increasing)* **d**-*limitwise monotonic degree on* $\mathcal{D}$ if there is a set $S \in \mathbf{a}$ that is a (support increasing / support strictly increasing) **d**-limitwise monotonic set on $\mathcal{D}$.

The function $f(x, s)$ is a *(support increasing / support strictly increasing)* **d**-*limitwise monotonic approximation on* $\mathcal{D}$ *for* S.

Definition 2.3. Let $\mathcal{D}$ be a fixed computable presentation of a computable linear order. Denote by $\mathbf{LM}^{\mathbf{d}}(\mathcal{D})$ the class of **d**-limitwise monotonic sets on $\mathcal{D}$; denote by $\mathbf{SILM}^{\mathbf{d}}(\mathcal{D})$ the class of support increasing **d**-limitwise monotonic sets on $\mathcal{D}$; and denote by $\mathbf{SSILM}^{\mathbf{d}}(\mathcal{D})$ the class of support strictly increasing **d**-limitwise monotonic sets on $\mathcal{D}$.

Definition 2.2 is a generalization of a definition introduced by Khisamiev in [7] to an arbitrary domain rather than ω. For simplicity, we state and prove all results for the special case when $\mathbf{d} = \mathbf{0}$; we also omit reference to the degree **d**. This causes no harm as all of our results relativize in the obvious straightforward manner. For notational clarity, we will maintain the convention of using uppercase characters (i.e., F, G, Φ) for limit functions and lowercase characters (i.e., f, g, φ) for approximation functions.

It is easy to see that the limitwise monotonic sets on an arbitrary computable linear order $\mathcal{D}$ coincide exactly with the limitwise monotonic sets on ω. We thus speak of *limitwise monotonic sets* rather than limitwise monotonic sets on $\mathcal{D}$ for some $\mathcal{D}$. This ambiguity is not possible, as we will show, when the requirement of being support increasing or support strictly increasing is added.

If $\mathcal{D}$ is a linear order, we denote by $\mathcal{D}^*$ the reverse linear order. Thus, for example, ω^* denotes the order type of the negative integers. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are linear orders, we denote by $\mathcal{D}_1 \cdot \mathcal{D}_2$ the product of $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e., $\mathcal{D}_2$ many copies of $\mathcal{D}_1$ whose points are indexed by a pair (u, v), with $u \in D_2$ giving the copy of D_1 and $v \in D_1$ giving the location within the copy of D_1.

3. Limitwise Monotonic Sets and Degrees

Before our study of limitwise monotonic sets on computable linear orders, we review literature results on limitwise monotonic sets and degrees and extend this body of work. Previous work has noted various constraints can be placed on the limitwise monotonic function. For example, the following result of Harris gives that any limitwise monotonic set has an injective limitwise monotonic function.

Theorem 3.1 ([5]). *If A is an infinite limitwise monotonic set, then the witnessing limitwise monotonic function can be chosen to be injective.*

Another known result is that one can pass from a limitwise monotonic set to a subset that is support strictly increasing limitwise monotonic on ω.

Theorem 3.2. *If A is an infinite limitwise monotonic set, then some $B \subseteq A$ is a support strictly increasing limitwise monotonic set on ω.*

Other results aim towards understanding which sets are limitwise monotonic. Since every limitwise monotonic set is Σ^0_2, the following two contrasting results situate the limitwise monotonic sets within the Turing degrees.

Theorem 3.3 ([5]; [13]). *Every Σ^0_2 degree contains a limitwise monotonic set.*

Theorem 3.4 ([10]). *There is a Δ^0_2 set that is not limitwise monotonic.*

Beyond noting that the non-limitwise monotonic Δ^0_2 set can be made low ([6]), the interaction between limitwise monotonic sets and other complexity classes has not been well-studied. Hirschfeldt, R. Miller, and Podzorov suggest that there are 1-random sets that are limitwise monotonic (see [6]). We demonstrate a much stronger result that yields this as a corollary.

Theorem 3.5 (J. Miller). *If $P \subseteq 2^\omega$ is any Π^0_1 class containing no finite sets, then P contains a limitwise monotonic set.*

Proof. We identify a subset of ω with its characteristic function and a characteristic function with its path in 2^ω. Letting $\{P_s\}_{s\in\omega}$ be a computable sequence of clopen sets such that $P = \bigcap_{s\in\omega} P_s$, we construct an infinite path σ in P using infinitely many strategies. The $(i+1)$st strategy at stage s will compute a finite approximation σ^{i+1}_s of σ extending σ^i_s by some non-zero length and will set the values of the limitwise monotonic approximation on part of its domain.

Strategy $i+1$: The action by Strategy $i+1$ at stage $s+1$ is as follows:

(1) Find the least $k \geq \max\left\{\left|\sigma^i_{s+1}\right|, \left|\sigma^{i+1}_s\right|\right\}$ such that there exists $\tau \in P_s$ properly extending σ^i_{s+1} with $|\tau| = k+1$ and $\tau(k) = 1$.
(2) Choose σ^{i+1}_{s+1} as the lexicographically least of all such τ of length $k+1$.
(3) If $\sigma^{i+1}_{s+1} = \sigma^{i+1}_s$, define $A^i_{s+1} = A^i_s$. Put $f(x, s+1) = f(x, s)$ for all $x \in A^i_s \cup \{2(i+1)\}$.
Otherwise, let $\{c_0 < c_1 < \cdots < c_{n-1}\}$ be the set of all c such that $|\sigma^i_{s+1}| \leq c < k$ and $\sigma^{i+1}_{s+1}(c) = 1$. Let $\{a^i_0, a^i_1, \ldots, a^i_{n-1}\}$ consist of the next n odd numbers not already in the support of F. Define $A^i_{s+1} = A^i_s \cup \{a^i_0, a^i_1, \ldots, a^i_{n-1}\}$. Put $f(x, s+1) = k$ for all $x \in A^i_s \cup \{2(i+1)\}$ and put $f(a^i_j, s+1) = c_j$ for all $0 \leq j < n$.

Construction: At stage s, Strategy i runs for each $i < s$ as described. When Strategy i is first run, it is initialized with $\sigma^i_s = \epsilon$ and $A^i_s = \emptyset$, where ϵ is the empty string.

Verification: Since P_s was clopen, the extension σ^{i+1}_{s+1} must exist. A simple argument by induction demonstrates that $\sigma^{i+1} := \lim_s \sigma^{i+1}_s$ exists. As $\sigma^i \subsetneq \sigma^{i+1}$ and

$\sigma^i \in P$ for all i, we have that $\sigma := \lim_i \sigma^i$ is an infinite path in P. Moreover, the function F is a limitwise monotonic function for the characteristic function of this path. □

As there are Π^0_1 classes containing only 1-random sets or DNR_2 sets, for example, we immediately obtain the following corollaries.

Corollary 3.6. *There is a* 1*-random set A that is limitwise monotonic.*

Corollary 3.7. *There is a DNR_2 set A that is limitwise monotonic.*

4. Domain Dependence

When studying support increasing (support strictly increasing) limitwise monotonic functions on computable domains, the smallest non-trivial domain choice is ω. At the opposite extreme are the rationals $\mathbb{Q}$. In between, of course, are a myriad of domain choices. Certain domain transitions introduce new sets whereas others introduce no new sets. Before studying these transitions (Section 4.1 and Section 4.2), we begin with propositions that apply in a much more general context. As both are easy, we defer the proofs to the reader.

Proposition 4.1. *Let $\mathcal{D} = (D : \prec)$ be a computable linear order. Then there is a computable enumeration $\{\varphi_i\}_{i \in \omega}$ of total computable functions $\varphi_i : D \times \omega \to \omega$ containing an approximation to every limitwise monotonic function on $\mathcal{D}$ and with the property that $\varphi_i(d,s) \leq \varphi_i(d,t)$ whenever $s < t$.*

Moreover, if containing approximations to the support increasing (support strictly increasing) limitwise monotonic functions is sufficient, the sequence can be chosen so that φ_i satisfies $\varphi_i(d,s) \leq \varphi_i(d',s)$ ($\varphi_i(d,s) < \varphi_i(d',s)$) whenever $d \prec d'$.

Proposition 4.2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be computable linear orders such that $\mathcal{D}_1$ computably embeds into $\mathcal{D}_2$. If S is a support increasing (support strictly increasing) limitwise monotonic set on $\mathcal{D}_1$, then S is a support increasing (support strictly increasing) limitwise monotonic set on $\mathcal{D}_2$.*

We also show that the support increasing (support strictly increasing) limitwise monotonic sets are a proper subset of the limitwise monotonic sets. The proof, though injury-free, contains key ideas that will manifest themselves throughout the paper.

Theorem 4.3. *The classes* $\mathbf{SILM}(\mathbb{Q})$ *and* $\mathbf{LM}$ *satisfy* $\mathbf{SILM}(\mathbb{Q}) \subsetneq \mathbf{LM}$.

Proof. We build an approximation function $f : \omega \times \omega \to \omega$ to a limitwise monotonic set A that diagonalizes against all support increasing limitwise monotonic functions on $\mathbb{Q}$. We fix a computable enumeration $\{\varphi_i\}_{i \in \omega}$ of total computable functions with domain $\mathbb{Q} \times \omega$ as in Proposition 4.1, with the property $\varphi_i(d,s) \leq \varphi_i(d',s)$ whenever $d \prec d'$.

We describe the general strategy to defeat an individual function φ_i, i.e., to assure that φ_i is not a support increasing limitwise monotonic approximation on $\mathbb{Q}$ for A.

Strategy i: The general strategy to ensure that A is not the range of Φ_i is as follows:

(1) Set a counter $k = k_i$ to zero.
(2) Put $f(\langle i, k\rangle, s) = 2i$.
(3) Wait for $\varphi_i(q,s) = 2i$ for some $q \in \mathbb{Q}$.

(4) Put $f(\langle i, k\rangle, s) = 2i + 1$.
(5) Wait for $\varphi_i(q, s) = 2i + 1$.
(6) Increment the counter k and return to Step 2.

Construction: At stage 0, define $f(x, s) = 0$ for all $x \in \omega$. At stage $s + 1$, start working to satisfy Strategy s as described. Define the value of $f(x, s + 1)$ to be $f(x, s)$ on any integer x for which the value of $f(x, s+1)$ is not otherwise explicitly defined by an active strategy.

Verification: Since there is no injury in the construction, it suffices to note that every strategy succeeds. There are several possible outcomes for Strategy i.

(a) The strategy spends cofinitely many stages waiting at Step 3. Then $2i \in A$ and $2i \notin \text{range}(\Phi_i)$.
(b) The strategy spends cofinitely many stages waiting at Step 5. Then $2i \notin A$ and $2i \in \text{range}(\Phi_i)$.
(c) The strategy visits Step 6 infinitely often. Then Φ_i is not a support increasing limitwise monotonic function on $\mathbb{Q}$, as the rationals q found in Step 3 form an infinite decreasing sequence, and thus the support does not have order type ω.

As Strategy i succeeds in all cases, we conclude that $f(x, s)$ is a limitwise monotonic set defeating every φ_i. Thus the range of F suffices to demonstrate the inclusion is strict. □

By Proposition 4.2, it follows that the class of support increasing (support strictly increasing) limitwise monotonic sets (on any domain) is a proper subset of the limitwise monotonic sets.

4.1. **Domain Transitions Increasing the Class of Sets.** The well-orders introduce new support increasing (support strictly increasing) limitwise monotonic sets when transitioning from a well-order to a suitable larger well-order.

Theorem 4.4. *For any computable ordinal α, there is a set $A \in$ **SSILM**$(\omega^{\alpha+1})$ with $A \notin$ **SILM**(ω^α).*

Proof. We build an approximation function $f : \omega^{\alpha+1} \times \omega \to \omega$ to a support strictly increasing limitwise monotonic set A on $\omega^{\alpha+1}$ that diagonalizes against all support increasing limitwise monotonic approximations on ω^α. We fix a computable enumeration $\{\varphi_i\}_{i\in\omega}$ of partial computable functions with domain $\omega^\alpha \times \omega$ as in Proposition 4.1.

We describe the general strategy to defeat an individual function φ_i, i.e., to assure that φ_i is not a support increasing limitwise monotonic approximation on ω^α for A.

Strategy i: The general strategy to ensure that A is not the range of Φ_i is as follows:

(1) Choose x *large*, i.e., two greater than any number already used in the construction. Put $f(\omega^\alpha(x + 1), s) = x$.
(2) As s increases, search for an ordinal β_0 such that $\varphi_i(\beta_0, s) = x$.
(3) Upon finding an ordinal β_0 such that $\varphi_i(\beta_0, s) = x$, switch to putting $f(w^\alpha(x + 1), s) = x + 1$.
(4) As s increases, wait for $\varphi_i(\beta_0, s) \geq x + 1$.
(5) Upon finding $\varphi_i(\beta_0, s) \geq x + 1$, put $f(\omega^\alpha(x) + \beta_0 + 1, s) = x$.

(6) As s increases, search for an ordinal $\beta_1 < \beta_0$ such that $\varphi_i(\beta_1, s) = x$.
(7) Upon finding an ordinal $\beta_1 < \beta_0$ with $\varphi_i(\beta_1, s) = x$, increase by one the value of $f(\gamma, s)$ for all ordinals $\gamma > \omega^\alpha(x)$ already in the support. Reset Strategy j for all active $j > i$ and return to Step 4 with β_1 and $\omega^\alpha(x)+\beta_0+1$ assuming the role of β_0 and $\omega^\alpha(x+1)$, respectively.

Construction: At stage 0, define $f(\beta, 0) = 0$ for all $\beta \in \omega^{\alpha+1}$. At stage $s+1$, start working to satisfy Strategy s as described. Define the value of $f(\beta, s+1)$ to be $f(\beta, s)$ on any ordinal β for which the value of $f(\beta, s+1)$ is not otherwise explicitly defined by an active strategy.

Verification: We verify the success of the construction in a sequence of claims.

Claim 4.4.1. Strategy i reaches Step 7 at most finitely often.

Proof. We assume Strategy j reaches Step 7 at most finitely often for all $j < i$ and show that Strategy i reaches Step 7 at most finitely often. From the inductive hypothesis, we have Strategy j reaches Step 7 only finitely many times before the last time it is itself reset; we therefore can assume it will never be reset again. The claim then follows for if Step 7 were reached infinitely often, there would be an infinite descending sequence in ω^α, an impossibility. □

Claim 4.4.2. The function f is a support strictly increasing limitwise monotonic approximation on $\omega^{\alpha+1}$.

Proof. That f is a total, computable function is immediate. By construction, the function f also satisfies $f(\beta, s) \leq f(\beta, s+1)$ for all $\beta \in \omega^{\alpha+1}$ and $s \in \omega$. We note that $F(\beta) = \lim_s f(\beta, s)$ exists for every ordinal $\beta \in \omega^{\alpha+1}$ as a consequence of Claim 4.4.1 together with the fact that if β is in the support of f at stage s, then any strategy initialized after stage s will effect the value of f only at ordinals $\beta' > \beta$.

We argue by induction on s that f is support strictly increasing at every stage by considering separately the cases when the value of $f(\gamma, s)$ changes for some ordinal γ. A new ordinal can appear in the support of f by Step 1 or Step 5; in either case, the choice of x large assures that f is support strictly increasing at stage $s+1$ if it was at stage s. The value of f can increase by Step 3; the choice of x large by other strategies again assures that f is support strictly increasing at stage $s+1$ if it was at stage s. The value of f can increase by one at many ordinals by Step 7, but since this happens on an end segment of the support, again f is support strictly increasing at stage $s+1$ if it was at stage s. □

Claim 4.4.3. Strategy i successfully defeats φ_i.

Proof. There are several possible outcomes for Strategy i, noting that Strategy i cannot reach Step 7 infinitely often by Claim 4.4.1.

(a) The strategy waits at Step 2 forever. Then $x \in A$ and $x \notin \text{range}(\Phi_i)$.
(b) The strategy waits at Step 4 forever. Then $x \notin A$ and $x \in \text{range}(\Phi_i)$.
(c) The strategy waits at Step 6 forever. Then $x \in A$ and either $x \notin \text{range}(\Phi_i)$ or Φ_i is not a support increasing limitwise monotonic function on ω^α.

As Strategy i succeeds in all cases, we conclude that φ_i cannot witness that S is a support increasing limitwise monotonic set on ω^α. □

From the claims, we conclude that $f(\beta, s)$ is a support strictly increasing limitwise monotonic approximation on $\omega^{\alpha+1}$ defeating every φ_i. Thus taking A to be the range of F suffices. $\square$

Remark 4.5. If we only wish a set $A \in \mathbf{SILM}(\omega^{\alpha+1})$ with $A \notin \mathbf{SILM}(\omega^\alpha)$, the argument can be altered to be injury-free. Rather than increase the value of $f(\gamma, s)$ for appropriate γ, we simply have Strategy i operate with the values of $2i$ and $2i+1$.

If we instead wish a set $A \in \mathbf{SSILM}(\omega^\gamma)$ with $A \notin \mathbf{SILM}(w^\alpha)$ for any $\alpha < \gamma$, it suffices to take the computable enumeration $\{\varphi_i\}_{i\in\omega}$ to include all total computable approximation functions on $\omega^\alpha \times \omega$ for all $\alpha < \gamma$.

We state the latter part of the remark as a corollary, adding an additional observation as well.

Definition 4.6. If γ is a limit ordinal, let $\mathcal{D}_\gamma$ be the linear order

$$\mathcal{D}_\gamma := \sum_{\alpha<\gamma^*} \omega^\alpha = \cdots + \omega^\alpha + \cdots + \omega^2 + \omega + 1.$$

Corollary 4.7. *If γ is a computable limit ordinal, then*

$$\cup_{\alpha<\gamma}\mathbf{SILM}(\omega^\alpha) = \mathbf{SILM}(\mathcal{D}_\gamma) \subsetneq \mathbf{SILM}(\omega^\gamma)$$

and

$$\cup_{\alpha<\gamma}\mathbf{SSILM}(\omega^\alpha) = \mathbf{SSILM}(\mathcal{D}_\gamma) \subsetneq \mathbf{SSILM}(\omega^\gamma).$$

Proof. The equalities follow from Proposition 4.2 and the fact that any support increasing (support strictly increasing) function on $\mathcal{D}_\gamma$ has support on an ordinal less than ω^γ. The proper containments follow similarly with the comments in Remark 4.5. $\square$

4.2. **Domain Transitions Not Increasing the Class of Sets.** Having shown that new support increasing (support strictly increasing) limitwise monotonic sets are introduced when transitioning from well-orders to sufficiently larger well-orders, we turn to demonstrating that there is no need to transition to $\mathbb{Q}$ to obtain all support increasing (support strictly increasing) limitwise monotonic sets on a computable domain.

Theorem 4.8. *The equalities*

$$\mathbf{SILM}(\omega^* \cdot \omega) = \mathbf{SILM}(\mathbb{Q}) \text{ and } \mathbf{SSILM}(\omega^* \cdot \omega) = \mathbf{SSILM}(\mathbb{Q})$$

hold.

Proof. The forward inclusions follows from Proposition 4.2. For the reverse inclusions, let $f(r, s)$ be a support increasing (support strictly increasing) limitwise monotonic approximation on $\mathbb{Q}$ for S. We will define a computable function $g : (\omega^* \cdot \omega) \times \omega \to \omega$ that will be a support increasing (support strictly increasing) limitwise monotonic approximation on $\omega^* \cdot \omega$ for S.

We view the domain $\omega^* \cdot \omega$ of G as being ω many copies of ω^* whose points are indexed by a pair (u, v), with $u \in \mathbb{Z}_{\geq 0}$ giving the copy of ω^* and $v \in \mathbb{Z}_{\leq 0}$ giving the location within the copy of ω^*.

The idea is to approximate the support of F. When a rational $r \in \mathbb{Q}$ appears in the support, we assign it to a point in $(\omega^* \cdot \omega)$ in an order-preserving manner. More specifically, if the rational number r appearing in the support is greater than all rationals already known to be in the support, we pick a large number u and assign r to $(u, 0)$; otherwise $r < r'$ for some rational r' already known to be in the support. We assign r to the pair immediately to the left of the point to which r' was assigned, sliding assignments to the left if there are any conflicts.

In order to keep track of the assignments between rationals q and pairs (u, v), we construct partial functions $h_s : \mathbb{Q} \to (\omega^* \cdot \omega)$ with domain T_s at stage s.

Construction: Fix an effective enumeration $\mathbb{Q} = \{r_n\}_{n\in\omega}$ of the rational numbers.

At stage 0, we define $T_0 = \emptyset$ and $h_0 : T_0 \to (\omega^* \cdot \omega)$ as the empty function. At stage $s+1$, we consider the set $T_{s+1} = \{r_n : n \leq s \text{ and } f(r_n, s) > 0\}$. If $T_{s+1} = T_s$, we let $h_{s+1} = h_s$.

Otherwise, we let $T_{s+1} \backslash T_s = \{r_{i_1} < r_{i_2} < \cdots < r_{i_n}\}$. In n substages, one for each rational r_{i_j}, we extend the domain of h_s to include r_{i_j} by making use of temporary auxiliary functions h^j_{s+1}. We let $h^0_{s+1} = h_s$. At substage j for $1 \leq j \leq n$, we define a function h^j_{s+1} with domain $T_s \cup \{r_{i_1}, \ldots, r_{i_j}\}$.

(1) If $r_{i_j} > r$ for every rational r in the domain of h^{j-1}_{s+1}, we let u be largest such that (u, v) is in the range of h^{j-1}_{s+1} for some v. We then define $h^j_{s+1}(r_{i_j}) = (u + 1, 0)$ and $h^j_{s+1}(r) = h^{j-1}_{s+1}(r)$ otherwise.

(2) Otherwise, we let r' be least so that $r_{i_j} < r' \in \mathrm{dom}(h^{j-1}_{s+1})$, say $h^{j-1}_{s+1}(r') = (u, v)$. We define $h^j_{s+1}(r_{i_j}) = (u, v - 1)$; then, for each $v'' \leq v - 1$ where $h^{j-1}_{s+1}(r'') = (u, v'')$ has been defined for some rational $r'' \in \mathrm{dom}(h^{j-1}_{s+1})$, we let $h^j_{s+1}(r'') = (u, v'' - 1)$. Since h^{j-1}_{s+1} has finite domain, this process terminates. For all other $r \in \mathrm{dom}(h^{j-1}_{s+1})$, we let $h^j_{s+1}(r) = h^{j-1}_{s+1}(r)$.

After the n substages are complete, we define $h_{s+1} = h^n_{s+1}$. Note that h_{s+1} is order-preserving and $\mathrm{dom}(h_s) \subseteq \mathrm{dom}(h_{s+1})$. For each rational $r \in T_{s+1}$, we define $g(h_{s+1}(r), s+1) = \max\{f(r, s+1), g(h_{s+1}(r), s)\}$; on all other rationals r, we define $g(r, s + 1) = 0$.

Verification: We begin by noting that, assuming $G((u, v))$ exists for all (u, v), the function $g((u, v), s)$ is a limitwise monotonic approximation on $\omega^* \cdot \omega$ and that each h_s is order-preserving by construction. We consider the set $T = \bigcup T_s$.

Claim 4.8.1. For any rational $r \in T$, $\lim_s h_s(r)$ exists. Moreover, for each $(u, v) \in \omega^* \cdot \omega$, there exist at most finitely many rationals r with $h_s(r) = (u, v)$ for some s.

Proof. We show that $h_s(r)$ and $h_{s+1}(r)$ cannot differ more than finitely often. We note that if $h_s(r) = (u, 0)$ for some u, then $h_t(r) = (u, 0)$ for all $t \geq s$, and so $\lim_s h_s(r)$ exists. We therefore assume that $h_s(r) = (u, v)$ with $v \neq 0$ and suppose $h_s(r) \neq h_{s+1}(r)$. In order for this to be the case, there must be a rational $q \in T_{s+1} \backslash T_s$ with $r < q < h_s^{-1}((u, 0))$ in the support of F. Since F is support increasing, the set $\{r \in \mathbb{Q} : r < q \text{ and } r \in \mathrm{supp}(F)\}$ is finite for each $q \in \mathbb{Q}$. Thus $\lim_s h_s(r)$ exists.

By a similar argument, for each $(u, v) \in \omega^* \cdot \omega$, there exist at most finitely many rationals r with $h_s(r) = (u, v)$ for some s. □

As a consequence of Claim 4.8.1 and F being support increasing, we can define $h(r) = \lim_s h_s(r)$ and $G(x) = \lim_s g(x, s)$, with these limits existing for all rationals r and pairs $(u, v) \in \omega^* \cdot \omega$.

We continue with a claim asserting that $G \circ h = F$ on the domain of h.

Claim 4.8.2. For any rational $r \in T$, we have $G(h(r)) = F(r)$.

Proof. Fixing a rational $r \in T$, by Claim 4.8.1, there is a least stage $\hat{s}$ satisfying $h_{\hat{s}}(r) = h(r)$. We note it follows that $h_s(r) = h(r)$ for all $s \geq \hat{s}$; denote this common value by (u, v). As a consequence of the definition of $g((u, v), s)$, it suffices to show the inequality $g((u, v), \hat{s} - 1) \leq F(r)$. If $(u, v) \in \text{range}(h_{\hat{s}-1})$, then $g((u, v), \hat{s} - 1)$ is $f(q, t)$ for some $q < r$ and $t \leq \hat{s}$. As $f(q, t) \leq F(q)$ since f is increasing in the second coordinate and $F(q) \leq F(r)$ since F is support increasing and $F(r) \notin \{0, 1\}$, we have the desired inequality. If instead $(u, v) \notin \text{range}(h_{\hat{s}-1})$, then $g((u, v), \hat{s} - 1) = 0$ and the desired inequality holds. Thus $g((u, v), \hat{s} - 1) \leq F(r)$, and so $G(r) = G((u, v)) = F(r)$. □

As $g(x, s) = 0$, and therefore $G(x) = 0$, whenever $x \notin \text{range}(h)$, we conclude that g is a support increasing (support strictly increasing) limitwise monotonic approximation on $\omega^* \cdot \omega$ for S. □

Though unnecessary for Theorem 4.8, we make the observation that the computable embedding of $\omega^* \cdot \omega$ into the rationals $\mathbb{Q}$ can be made to be cofinal in $\mathbb{Q}$, yielding the following as a corollary.

Corollary 4.9. *If $S \subseteq \omega$ is a support increasing (support strictly increasing) limitwise monotonic set on $\mathbb{Q}$, then there is a support increasing (support strictly increasing) limitwise monotonic set on $\mathbb{Q}$ whose support is cofinal in $\mathbb{Q}$.*

Finally, as a corollary to Theorem 4.8, we have that $\omega^* \cdot \omega$ is a minimal domain $\mathcal{D}$ with the property that $\mathbf{SILM}(\mathcal{D}) = \mathbf{SILM}(\mathbb{Q})$ or $\mathbf{SSILM}(\mathcal{D}) = \mathbf{SSILM}(\mathbb{Q})$. We recall that linear orders $\mathcal{D}_1$ and $\mathcal{D}_2$ are *equimorphic* if $\mathcal{D}_1$ order embeds into $\mathcal{D}_2$ and $\mathcal{D}_2$ order embeds into $\mathcal{D}_1$.

Theorem 4.10. *Let $\mathcal{D}$ be a computable linear order which order embeds into $\omega^* \cdot \omega$. If $\mathbf{SILM}(\mathcal{D}) = \mathbf{SILM}(\mathbb{Q})$ or $\mathbf{SSILM}(\mathcal{D}) = \mathbf{SSILM}(\mathbb{Q})$, then $\mathcal{D}$ is equimorphic with $\omega^* \cdot \omega$.*

Proof. If $\mathcal{D}$ order embeds into $\omega^* \cdot \omega$, then $\mathcal{D}$ is of one of the following forms: m, $m + \omega^* \cdot n$, $m + \omega^* \cdot n + \omega$, or $m + \omega^* \cdot \omega$, where m and n are integers. The first and second have no increasing ω sequence, and thus admit no support increasing (support strictly increasing) functions. For the third, all but finitely much of the support of any support increasing (support strictly increasing) function must live in the ω, and thus $\mathbf{SILM}(\mathcal{D}) = \mathbf{SILM}(\omega) \neq \mathbf{SILM}(\mathbb{Q})$ ($\mathbf{SSILM}(\mathcal{D}) = \mathbf{SSILM}(\omega) \neq \mathbf{SSILM}(\mathbb{Q})$). Finally, any linear order of the form $m + \omega^* \cdot \omega$ is equimorphic with $\omega^* \cdot \omega$. □

5. Limitwise Monotonic Functions on $\mathbb{Q}$

Having compared the class of support increasing (support strictly increasing) limitwise monotonic sets for various domains, we turn to studying these classes for specific domains, beginning with $\mathbb{Q}$. The support increasing limitwise monotonic sets and degrees on $\mathbb{Q}$ and the support strictly increasing limitwise monotonic sets and degrees on $\mathbb{Q}$ behave very differently. The former is closed under unions and joins, but the latter is not. Every Δ^0_2 degree is support increasing limitwise monotonic on $\mathbb{Q}$, but not every Δ^0_2 degree is support strictly increasing limitwise monotonic on $\mathbb{Q}$.

Theorem 5.1. *The class $\mathbf{SILM}(\mathbb{Q})$ is closed under finite unions and finite joins.*

Proof. Let A and B be support increasing limitwise monotonic sets on $\mathbb{Q}$ with limit approximation functions f_A and f_B, respectively. We define a support increasing limitwise monotonic approximation $f_{A\cup B}$ on $\mathbb{Q}$ for $A \cup B$ to show closure under (finite) union; for (finite) join, it suffices to observe that the sets $2A = \{2a : a \in A\}$ and $2B + 1 = \{2b + 1 : b \in B\}$ are support increasing limitwise monotonic sets on $\mathbb{Q}$, and thus so is $A \oplus B = 2A \cup (2B + 1)$.

The idea when defining $f_{A\cup B}$ is to add a new rational q_k in the support of $F_{A\cup B}$ in an order-preserving manner whenever a new rational q_i is found in the support of F_A. Then, at all future stages, we will have q_k associated with q_i so that $f_{A\cup B}(q_k, s) = f_A(q_i, s)$. When a new rational q_j is found in the support of F_B, a new rational q_k is added in the support of $F_{A\cup B}$ in an order-preserving manner and associated with q_j. However if the approximation $f_B(q_j, s)$ increases too quickly or too slowly, the rational q_k becomes associated with some element q_i in the support of F_A and a new rational q_k is added to the support of $F_{A\cup B}$ and associated with q_j.

As preparation, fix effective enumerations of the rationals numbers $\mathbb{Q}_A = \{q_i\}_{i\in\omega}$, $\mathbb{Q}_B = \{q_j\}_{j\in\omega}$, and $\mathbb{Q}_{A\cup B} = \{q_k\}_{k\in\omega}$. Note that we view the domains $\mathbb{Q}_A$ of f_A, $\mathbb{Q}_B$ of f_B, and $\mathbb{Q}_{A\cup B}$ of $f_{A\cup B}$ as disjoint. For notational convenience, we define $f_Z(q, s)$ to be $f_A(q, s)$ if $q \in \mathbb{Q}_A$ and $f_B(q, s)$ if $q \in \mathbb{Q}_B$. We denote the sets $\{q_i \in \mathbb{Q}_A : f_A(q_i, s) > 0 \text{ and } i \leq s\}$, $\{q_j \in \mathbb{Q}_B : f_B(q_j, s) > 0 \text{ and } j \leq s\}$, and $\{q_k \in \mathbb{Q}_{A\cup B} : f_{A\cup B}(q_k, s) > 0 \text{ and } k \leq s\}$ by $\operatorname{supp}(f_A, s)$, $\operatorname{supp}(f_B, s)$, and $\operatorname{supp}(f_{A\cup B}, s)$, respectively, and assume that at any stage s, these sets are finite.

In order to keep track of the association between rationals in the support of F_A, F_B, and $F_{A\cup B}$, auxiliary functions h_s for $s \in \omega$ are used.

Construction: At stage 0, we define $f_{A\cup B}(q_k, 0) = 0$ for all $q_k \in \mathbb{Q}_{A\cup B}$ and h_0 as the empty function.

At stage $s+1$, we act to define $f_{A\cup B}(q_k, s+1)$ for $q_k \in \operatorname{supp}(f_{A\cup B}, s)$ and to add rationals q to $\operatorname{supp}(f_{A\cup B})$ to represent rationals $q_i \in \operatorname{supp}(f_A, s+1)\backslash\operatorname{supp}(f_A, s)$ and $q_j \in \operatorname{supp}(f_B, s+1)\backslash\operatorname{supp}(f_B, s)$.

Begin by considering all $q_k \in \operatorname{supp}(f_{A\cup B}, s)$ such that $h_s(q_k) \in \mathbb{Q}_A$. For each such q_k, put $f_{A\cup B}(q_k, s+1) = f_A(h_s(q_k), s+1)$ and define $h_{s+1}(q_k) = h_s(q_k)$.

For each $q_k \in \operatorname{supp}(f_{A\cup B}, s)$ such that $h_s(q_k) \in \mathbb{Q}_B$, do the following:

- If there is a $q_\ell \in \operatorname{supp}(f_{A\cup B}, s)$ with $h_s(q_\ell) \in \mathbb{Q}_A$, $q_\ell < q_k$, and $f_A(h_s(q_\ell), s+1) > f_B(h_s(q_k), s+1)$, let q_0 be the greatest such q_ℓ. Put $f_{A\cup B}(q_k, s+1) = f_A(h_s(q_0), s+1)$ and define $h_{s+1}(q_k) = h_s(q_0)$. Choose m least such that $q_m \notin \operatorname{supp}(f_{A\cup B}, s)$, m has not been chosen by some other q_k this stage, and putting $f_{A\cup B}(q_m, s+1) = f_B(h_s(q_k), s+1)$ keeps $f_{A\cup B}$ order-preserving; put $f_{A\cup B}(q_m, s+1) = f_B(h_s(q_k), s+1)$, and define $h_{s+1}(q_m) = h_s(q_k)$.
- If there is a $q_\ell \in \operatorname{supp}(f_{A\cup B}, s)$ with $h_s(q_\ell) \in \mathbb{Q}_A$, $q_\ell > q_k$, and $f_A(h_s(q_\ell), s+1) < f_B(h_s(q_k), s+1)$, let q_0 be the least such q_ℓ. Put $f_{A\cup B}(q_k, s+1) = f_A(h_s(q_0), s+1)$ and define $h_{s+1}(q_k) = h_s(q_0)$. Choose m least such that $q_m \notin \operatorname{supp}(f_{A\cup B}, s)$, m has not been chosen by some other q_k this stage, and putting $f_{A\cup B}(q_m, s+1) = f_B(h_s(q_k), s+1)$ keeps $f_{A\cup B}$ order-preserving; put $f_{A\cup B}(q_m, s+1) = f_B(h_s(q_k), s+1)$, and define $h_{s+1}(q_m) = h_s(q_k)$.
- Otherwise, put $f_{A\cup B}(q_k, s+1) = f_B(h_s(q_k), s+1)$ and define $h_{s+1}(q_k) = h_s(q_k)$.

Towards representing rationals $q_i \in \text{supp}(f_A, s{+}1)\backslash\text{supp}(f_A, s)$ and $q_j \in \text{supp}(f_B, s{+}1)\backslash\text{supp}(f_B, s)$, add a new element for each in the support of $F_{A\cup B}$. More specifically, for each such q_i or q_j (hereout termed q), choose k least so that $q_k \notin \text{supp}(f_{A\cup B}, s)$ and putting $f_{A\cup B}(q_k, s{+}1) = f_Z(q, s{+}1)$ keeps $f_{A\cup B}$ order-preserving; put $f_{A\cup B}(q_k, s+1) = f_Z(q, s+1)$ and define $h_{s+1}(q_k) = q$.

Finally, for all $q_k \in \mathbb{Q}_{A\cup B}$ for which $f_{A\cup B}(q_k, s{+}1)$ has not already been defined, put $f_{A\cup B}(q_k, s+1) = 0$.

Verification: We demonstrate that $f_{A\cup B}$ is a support increasing limitwise monotonic approximation on $\mathbb{Q}$ for $A \cup B$. By construction, the approximation function $f_{A\cup B}$ is support increasing; its range at stage s is also the union of the ranges of f_A and f_B at stage s.

Also note that $h_s(q_k)$ can only change from an element of $\mathbb{Q}_B$ to an element of $\mathbb{Q}_A$, and thus can change at most once. Thus $\lim_s h_s(q_k)$ exists for all $q_k \in \mathbb{Q}_{A\cup B}$, and thus since $F_A(q_i)$ and $F_B(q_k)$ exist by hypothesis, $F_{A\cup B}(q_k) = \lim_s f_{A\cup B}(q_k, s) = \lim_s f_Z(h_s(q_k), s)$ exists for all q_k.

So $F_{A\cup B}$ is total and its range is contained in $A\cup B$. All that remains to show is that the reverse inclusion holds. For elements in A, this is clear. For $q_j \in \text{supp}(F_B)$ with $F_B(q_j) \in B\backslash A$, let a_0 be the greatest element of A less than $F_B(q_j)$ (if there is one), and a_1 be the least element of A greater than $F_B(q_j)$. Choose s_0 sufficiently large such that $f_A(a_0, s_0) = F_A(a_0)$, $f_A(a_1, s_0) = F_A(a_1)$ and $q_j \in \text{supp}(f_A, s_0)$. Then there is some q_k such that $h_{s_0}(q_k) = q_j$, and for all $s > s_0$, $h_s(q_k) = q_j$ (any element that forced a change in h_s would violate f_A being order preserving). Thus $F_{A\cup B}(q_k) = F_B(q_j)$. ☐

Unlike **SILM**($\mathbb{Q}$), the class **SSILM**($\mathbb{Q}$) is not closed under unions or joins.

Theorem 5.2. *The class* **SSILM**($\mathbb{Q}$) *is not closed under unions or joins.*

Proof. We construct support strictly increasing limitwise monotonic approximations f_A and f_B on ω for sets A and B so that $A \cup B \notin$ **SSILM**($\mathbb{Q}$), yielding a stronger result than stated. As the construction will have $A \subseteq \{2n : n \in \omega\}$ and $B \subseteq \{2n + 1 : n \in \omega\}$, this also establishes the lack of closure of **SSILM**($\mathbb{Q}$) under joins.

The construction will diagonalize against all candidate support strictly increasing limitwise monotonic functions Φ_i on $\mathbb{Q}$ for $A \cup B$. The basic idea is to choose an element in the support of φ_i with current value in A and use an element of B to force φ_i to increase its approximation. After φ_i matches this challenge by taking a value in B, an element in A is used to further force φ_i to increase its approximation. In this manner, the approximation value will be forced to infinity.

Strategy i will work to defeat Φ_i. The choice of the witness x used to do this requires discussion. For a given x chosen for this role, it is possible, by examining the currently active strategies and their work so far, to determine an upper bound m on the size of the supports of F_A and F_B below x. Basically, every element of the support of F_A gives rise to at most i many elements in the support of F_B when Strategy i encounters it, and vice versa. Where these new elements will appear depends on the arrangement of the elements of A and B. This process terminates as sufficiently many will appear above x.

The combinatorial specifics are unimportant, but the key observation is that m will not depend on x, and thus we can use m to influence our choice of x. In particular, we choose $x \geq 2^m$, and whenever we need a new element between two existing elements of the support, we choose a point midway between them. In this way, we need never worry about running out of points in the domain of f_A and f_B.

Further, for a given y, $f_A(y,\cdot)$ and $f_B(y,\cdot)$ will assume at most two nonzero values, while for the above x, $f_A(x,\cdot)$ will assume only one nonzero value. Thus $2m$ is an upper bound on the number of values of $A \cup B$ beneath $F_A(x)$. So if we set $F_A(x) \geq x^2 \geq 2^{2m}$, we can similarly never worry about running out of values in the range.

Strategy i: The action taken by Strategy i is as follows:

(1) Choose a large witness x. Let $u = x$ and $a \geq x^2$ be even, and set $f_A(u,s) = a$.
(2) Wait for a rational z to appear with $\varphi_i(z,s) = a$.
(3) Let $\{b_0 < b_1 < \cdots < b_n\} = B \upharpoonright a$. Choose v and an odd b with $b_n < b < a$, such that $f_B(v,s{+}1) = b$ will not violate support strictly increasing. Define $f_B(v,s+1) = b$.
(4) Protect B on the interval $[b,a]$ while waiting for the range of $\varphi_i(\cdot,s)$ to agree with $A \cup B$ on $[b,a]$.
(5) Let b' be the least element of B greater than a. Let a' be the greatest element of A less than b'. Choose an odd c between a' and b', and set $f_B(v,s+1) = c$.
(6) Protect $A \cup B$ on $[b,a]$ while waiting for $\varphi_i(z,s) = f_A(w,s)$ for some $w > u$ or $\varphi_i(z,s) = f_B(w,s)$ for some $w > v$.
(7) If $\varphi_i(z,s) = f_A(w,s)$, return to Step 3 using $u = w$ and $a = f_A(w,s)$. If $\varphi_i(z,s) = f_B(w,s)$, proceed to Step 8 using $v = w$ and $b = f_B(w,s)$.
(8) Let $\{a_0 < a_1 < \cdots < a_n\} = A \upharpoonright b$. Choose u and an even a with $a_n < a < b$, such that $f_A(u,s{+}1) = a$ will not violate support strictly increasing. Define $f_A(u,s+1) = a$.
(9) Protect A on $[a,b]$ while waiting for the range of $\varphi_i(\cdot,s)$ to agree with $A \cup B$ on $[a,b]$.
(10) Let a' be the least element of A greater than b. Let b' be the greatest element of B less than a'. Choose an even c between b' and a', and set $f_A(u,s+1) = c$.
(11) Protect $A \cup B$ on $[a,b]$ while waiting for $\varphi_i(z,s) = f_A(w,s)$ for some $w > u$ or $\varphi_i(z,s) = f_B(w,s)$ for some $w > v$.
(12) If $\varphi_i(z,s) = f_A(w,s)$, return to Step 3 using $u = w$ and $a = f_A(w,s)$. If $\varphi_i(z,s) = f_B(w,s)$, return to Step 8 using $v = w$ and $b = f_B(w,s)$.

Strategy i can have several possible outcomes:

w_2: Wait forever at Step 2. Then $a \in A \cup B$, while $a \notin \text{range } \Phi_i$.
w_4: Wait forever at Step 4. Then $(A \cup B) \upharpoonright [b,a] \neq \text{range } \Phi_i \upharpoonright [b,a]$.
w_6: Wait forever at Step 6. If $\Phi_i(z) = a$, then $(A \cup B) \upharpoonright [b,a]$ contains fewer elements than range $\Phi_i \upharpoonright [b,a]$. If $\Phi_i(z) \neq a$, then $\Phi_i(z) \notin A \cup B$.
w_9: Wait forever at Step 9. Then $(A \cup B) \upharpoonright [a,b] \neq \text{range } \Phi_i \upharpoonright [a,b]$.
w_{11}: Wait forever at Step 11. If $\Phi_i(z) = b$, then $(A \cup B) \upharpoonright [a,b]$ contains fewer elements than range $\Phi_i \upharpoonright [a,b]$. If $\Phi_i(z) \neq b$, then $\Phi_i(z) \notin A \cup B$.
∞: Return infinitely often to Steps 7 or 12. Then $\Phi_i(z) \uparrow$.

Unfortunately, there are several ways that strategies can injure each other. If strategy i wishes to add a value to an interval protected by a strategy $j < i$, strategy i is temporarily injured. It begins again with a new large x. If the interval in question ever becomes unprotected, it discards this latter attempt and returns to the current one.

If a Strategy i is attempting to protect an interval via Step 6, note that the protected interval will contain no elements of B. So no other strategy will feel compelled to add an element. Thus this protection results in no injuries. Similarly the protection at Step 11.

If a Strategy i is attempting to protect an interval via Step 4, note that the only steps at which another strategy might attempt to add an element to B are Steps 3 and 5. Because the only element of B in the interval is the left end-point, Step 5 can always choose an element outside the interval. Thus the only possible violation comes from another strategy at Step 3.

If a Strategy $j < i$ wishes to add an element to the interval via Step 3, strategy i is temporarily injured. It begins again with a new large x. If Strategy j ever reaches Step 5, we require that the c chosen be outside the protected interval, thus allowing strategy i to discard the latter attempt and return to the current one.

This requirement, however, can result in further injury. Suppose the interval $[b, a]$ is being protected by Strategy i, and Strategies $j_0 < j_1 < \cdots < j_n$ have placed elements $b_0, b_1, \ldots, b_n$ into the interval. When Strategy j_k reaches Step 5, if b_k is the rightmost element of B in the interval, it can proceed as normal, and the result will be that the new c will be chosen outside the interval.

If it is not the rightmost element of B, then what we do next depends on if there is an $l < k$ with $b_k < b_l$. If so, Strategy k is injured until such time as there is not, beginning again with a new large x. If all the $b_l > b_k$ come from Strategies $j_l > j_k$, then they are pushed out of the way. Each such j_l is forced to choose a c_l outside the interval and set $f_B(v_l, s+1) = c_l$. Strategy j_k then proceeds with Step 5, and then the affected j_l return to Step 3 (which may result in immediate injury to them, as j_k is now protecting an interval).

Verification: Observe that any injury to Strategy i that is not eventually resolved can only be the result of some Strategy $j < i$ attaining one of w_4, w_6, w_9 or w_{11}. So once all Strategies $j < i$ which are going to attain a finite win do so, some instance of i will be able to act infinitely often, and thus will reach one of the outcomes. Thus Φ_i will be defeated. □

Having shown that the classes **SILM**($\mathbb{Q}$) and **SSILM**($\mathbb{Q}$) behave differently with respect to closure under unions and joins, we turn to showing that they also behave differently with respect to representing the Δ^0_2 degrees.

Theorem 5.3. *Every Δ^0_2 degree contains a support increasing limitwise monotonic set on $\mathbb{Q}$.*

Proof. The idea, just as in the proof that every Δ^0_2 degree contains a limitwise monotonic set on ω (see [5] and [13]), is to show that $S \oplus \omega$ is a support increasing limitwise monotonic set on $\mathbb{Q}$ for any S in a Δ^0_2 degree. Towards this end, fix a Δ^0_2 degree $\mathbf{d}$, a set $S \in \mathbf{d}$, and a Δ^0_2 approximation $\{S_s\}_{s\in\omega}$ to S with $S_0 = \emptyset$ and $|S_s| < \infty$ for all s.

We define a function $f : \mathbb{Q} \times \omega \to \omega$ that will serve as a support increasing limitwise monotonic approximation on $\mathbb{Q}$ for the set $S \oplus \omega \in \mathbf{d}$. The idea will be to use a sequence of rational numbers (of order type ω) to put $2n + 1$ in the range of F for all n and to use the intervals in between these rationals to put $2n$ in the range of F if n appears to be in S. When n appears to leave S, we remove $2n$ from the range by increasing the value of f to $2n+1$.

Construction: We construct a support increasing limitwise monotonic approximation on $\mathbb{Q}$ in ω many stages by defining $f(q,s)$ for all rationals q at stage s. As preparation, fix a computable enumeration $\{q_i\}_{i\in\omega}$ of $\mathbb{Q}\backslash\mathbb{N}$. Let $\lceil q_i \rceil$ denote the least non-negative integer greater than q_i.

At stage 0, we set $f(n,0) = 2n+1$ for all $n \in \mathbb{N}$ and $f(q,0) = 0$ for all $q \in \mathbb{Q}\backslash\mathbb{N}$. At stage $s+1$, we set $f(n,s+1) = 2n+1$ for all $n \in \mathbb{N}$. The definition of $f(q_i, s+1)$ for $q_i \in \mathbb{Q}\backslash\mathbb{N}$ depends on the values of $S_s(\lceil q_i \rceil)$ and $S_{s+1}(\lceil q_i \rceil)$.

1. If $S_{s+1}(\lceil q_i \rceil) = 0$, we set $f(q_i, s+1) = 2\lceil q_i \rceil + 1$ if q_i is in the support of f and $f(q_i, s+1) = 0$ otherwise.
2. If $S_{s+1}(\lceil q_i \rceil) = 1$ and $S_s(\lceil q_i \rceil) = 1$, we set $f(q_i, s+1) = f(q_i, s)$.
3. If $S_{s+1}(\lceil q_i \rceil) = 1$ but $S_s(\lceil q_i \rceil) = 0$, we set $f(q_i, s+1) = 2\lceil q_i \rceil$ if i is the minimal integer j with $\lceil q_j \rceil = \lceil q_i \rceil$ and q_j not in the support of f; otherwise, we set $f(q_i, s+1) = f(q_i, s)$.

This completes the construction.

Verification: We note that by construction, we have $F(n) = 2n+1$ for all $n \in \mathbb{N}$ and $f(q_i, s) \leq f(q_i, s+1)$ and $f(q_i, s) \in \{0, 2\lceil q_i \rceil, 2\lceil q_i \rceil + 1\}$ for all $q_i \in \mathbb{Q}\backslash\mathbb{N}$. We therefore need only verify that the range of F is $S \oplus \omega$ and that every proper initial segment of the support is finite. Towards showing the former, we note that $\emptyset \oplus \omega$ is a subset of the range of F since $F(n) = 2n+1$ for all $n \in \mathbb{N}$. Also $S \oplus \emptyset$ is a subset of the range of F. For if $n \in S$, there is a stage t such that $n \notin S_t$ but $n \in S_s$ for all $s > t$. At stage $t+1$, Case 3 will set $f(q_i, t+1) = 2n$ for some rational q_i. Then $F(q_i) = 2n$ since we will never be in Case 1 again. If instead $n \notin S$, there is a least stage t such that $n \notin S_s$ for all $s \geq t$. If $t = 0$, then $2n$ is not in the range of F as we never reach Case 3; if $t > 0$, then $2n$ is not in the range of F as we will execute Case 1 at stage t and never execute Case 3 thereafter.

Finally, every proper initial segment of the support is finite as a consequence of $S(n) = \lim_s S_s(n)$ existing for all $n \in \omega$. More specifically, exactly one new rational $q_i \in \mathbb{Q}\backslash\mathbb{N}$ enters the support of F below an integer $k \in \mathbb{N}$ exactly when $S_{s+1}(\ell) = 1$ but $S_s(\ell) = 0$ for some $\ell \leq k$. However, there are only finitely many pairs (ℓ, s) with this property since $\{S_s\}_{s\in\omega}$ was a Δ^0_2 approximation. $\square$

In contrast, not every Δ^0_2 degree contains a support strictly increasing limitwise monotonic set on $\mathbb{Q}$.

Theorem 5.4. *There is a Δ^0_2 degree that is not support strictly increasing limitwise monotonic on $\mathbb{Q}$.*

Proof. We build a Δ^0_2 set A such that $B \notin \mathbf{SSILM}(\mathbb{Q})$ for any set B with $0 <_T B \leq_T A$. The construction will define a sequence of strings $\sigma_{k,s}$, with A being given by $\bigcup_k \sigma_k$ and $\sigma_k = \lim_s \sigma_{k,s}$. We fix a computable enumeration $\{\varphi_e\}_{e\in\omega}$ of total computable functions with domain $\mathbb{Q} \times \omega$ as in Proposition 4.1 and a computable enumeration $\{\Theta_i\}_{i\in\omega}$ of all Turing functionals.

It suffices for A to satisfying the following requirements.

Requirement $\mathcal{R}_{e,i}$: If $B := \Theta_i^A >_T 0$, then $B \neq \mathrm{range}(\Phi_e)$.

Requirement $\mathcal{P}_j$: That $A \neq \Theta_j^{\emptyset}$.

The strategy to meet Requirement $\mathcal{P}_j$ is standard: choose a large witness z, wait for $\Theta_j^{\emptyset}(z)$ to converge, and then define A so that $A(z) \neq \Theta_j^{\emptyset}(z)$. The strategy to meet Requirement $\mathcal{R}_{e,i}$ is more complex. Indeed, we use infinitely many Strategies $\mathcal{R}_{e,i,m}$ to assure its satisfaction. The idea is to repeatedly add and remove an element from B. Each time this is done, the approximation $\varphi_i(x, s)$ will need to increase for all x on which Φ_i is at least the given element, else it will cease to enumerate B. In this fashion we force $\Phi_i(x)$ to diverge for some x. Requirement $\mathcal{R}_{e,i,m}$ will be responsible for driving $\Phi_i(x)$ up by some amount. Splitting $\mathcal{R}_{e,i}$ into the $\mathcal{R}_{e,i,m}$ allows us to assure that A is Δ_2^0.

Since we don't have direct control over the set B, the strategy $R_{e,i}$ searches for $n \in \omega$ and $\sigma_a, \sigma_b \supset \sigma$ such that $n \in \Theta_i^{\sigma_a}$ and $n \notin \Theta_i^{\sigma_b}$. If B is noncomputable, infinitely many n will have such σs. We can then add and remove n from B by changing the initial segment of A between σ_a and σ_b.

Strategy $\mathcal{R}_{e,i,m}$: More specifically, given an initial segment σ of A, the strategy acts as follows:

(1) Search for σ_a and σ_b properly extending σ and $n \in \omega$ such that $\Theta_i^{\sigma_a}(n) \downarrow= 1$ and $\Theta_i^{\sigma_b}(n) \downarrow= 0$.
(2a) If $m = 0$, wait for an $x \in \mathbb{Q}$ to appear such that $\varphi_e(x, s) = n$.
(2b) If $m \neq 0$, let x be the x used by Strategy $\mathcal{R}_{e,i,0}$. Wait until $\varphi_e(x, s) \geq n$.
(3) Wait for $\varphi_e(x, s)$ to increase beyond n.
(4) Wait until some $\mathcal{R}_{e,i,m+1}$ node directly descended from the current node is waiting at Step 2b.
(5) Wait for some $y < x$ to appear with $\varphi_e(y, s) = n$.
(6) Wait for $\varphi_e(y, s)$ to increase beyond n.
(7) Return to Step 4.

Strategy $\mathcal{R}_{e,i,m}$ will have the following temporary outcomes.

$\sigma^\frown 0$: The strategy is searching at Step 1.
σ_a: The strategy is waiting at Step 2a, Step 2b, or Step 5.
σ_b: The strategy is waiting at Step 3, Step 4, or Step 6.

Strategy $\mathcal{P}_j$ will have the following temporary outcomes.

$\sigma^\frown 0$: The strategy is waiting for $\Theta_j^{\emptyset}(z)$ to converge, or $\Theta_j^{\emptyset}(z) \downarrow= 1$.
$\sigma^\frown 1$: $\Theta_j^{\emptyset}(z) \downarrow= 0$.

Construction: Arrange the strategies on a tree in the usual fashion. If α is a node at level k of the tree which is active at stage s, let $\sigma_{k,s}$ be the current outcome of α.

Verification: We verify the success of the construction by establishing that a true path exists and that every requirement is met.

Claim 5.4.1. Every strategy which acts infinitely often spends cofinitely many stages at some step.

Proof. Since the claim is clear for Strategies $\mathcal{P}_j$, it suffices to consider Strategies $\mathcal{R}_{e,i,m}$. For a fixed $\mathcal{R}_{e,i,m}$ strategy α, let N be the greatest n chosen by an $R_{e,i,m+1}$ strategy lower on the tree. Every time α reaches Step 7, some new $y < x$ has appeared with $\varphi_e(y,s) \in (n,N]$. This can happen at most $N-n$ times before $\varphi_e(x,s)$ is necessarily greater than N, at which point α will never leave Step 4. Thus α eventually settles. $\square$

Since a true path exists, for a strategy α along the true path, we can define the final outcome of α to be whatever temporary outcome it has cofinitely often. It follows that if α is at level k in the tree, for cofinitely many s the string $\sigma_{k,s}$ will be set by α. Thus $\sigma_k = \lim_s \sigma_{k,s}$ exists.

Claim 5.4.2. Every requirement is met.

Proof. Again the claim is clear for Requirements $\mathcal{P}_j$, so it suffices to consider Requirements $\mathcal{R}_{e,i}$. For a given e and i, if every $\mathcal{R}_{e,i,m}$ strategy along the true path waits forever at Step 4, then $\lim_s \varphi_e(x,s) = \infty$ for the x chosen by the Strategy $\mathcal{R}_{e,i,0}$.

Otherwise, let α be the earliest $\mathcal{R}_{e,i,m}$ strategy along the true path not waiting forever at Step 4. It cannot be waiting forever at Step 2b, as then the $\mathcal{R}_{e,i,m-1}$ strategy would be waiting at some step other than Step 4. If α waits forever at Step 1, then either Θ_i^A is partial or it can be computed from σ, and thus is computable. If α waits forever at Step 2a or Step 5, then $n \in \Theta_i^A$, but $n \notin \text{range}(\Phi_e)$. If α waits forever at Step 3 or Step 6, then $n \in \text{range}(\Phi_e)$, but $n \notin \Theta_i^A$. $\square$

Since the P requirements ensure that A is non-computable, the degree of A is our desired Δ_2^0 degree. $\square$

6. Limitwise Monotonic Functions on Well-Orders

Though the inclusions $\mathbf{SILM}(\omega^\alpha) \subsetneq \mathbf{SILM}(\mathbb{Q})$ and $\mathbf{SSILM}(\omega^\alpha) \subsetneq \mathbf{SSILM}(\mathbb{Q})$ are strict, the classes $\mathbf{SILM}(\omega^\alpha)$ and $\mathbf{SSILM}(\omega^\alpha)$ share some of the properties of the classes $\mathbf{SILM}(\mathbb{Q})$ and $\mathbf{SSILM}(\mathbb{Q})$. For example, the class $\mathbf{SILM}(\omega^\alpha)$ is closed under unions and joins for at least some ordinals α, whereas the class $\mathbf{SSILM}(\omega^\alpha)$ is never closed under unions and joins. The major difference is that there are Δ_2^0 degrees that are not $\mathbf{SILM}(\omega^\alpha)$.

Theorem 6.1. *The class* $\mathbf{SILM}(\omega^\gamma)$ *is closed under unions and joins if* γ *is a computable ordinal satisfying* $(\forall \lambda < \omega^\gamma)[\lambda^2 < \omega^\gamma]$.

By way of illustration, note that the ordinals 1 and ω have this property while the ordinal 2 does not.

Proof. Let γ be such a computable ordinal, and let $A, B \in \mathbf{SILM}(\omega^\gamma)$ have limit approximation functions f_A and f_B. As in Theorem 5.1, we construct a limit approximation function $f_{A\cup B} : \omega^\gamma \times \omega \to \omega$ for $A \cup B$. Again the idea is the same, though the implementation is more complicated. Ordinals in the support of F_A are placed in the support of $F_{A\cup B}$ sufficiently spread apart so that there is room to insert the elements of B in an appropriate place.

As preparation, let $\{\alpha_i\}_{i\in\omega}$ and $\{\beta_i\}_{i\in\omega}$ be computable strictly increasing sequences of ordinals in the support of F_A and F_B, respectively. Note that these sequences are necessarily cofinal in the support. Let $c_i = \min\{F_A(\alpha_i), F_B(\beta_i)\}$. The definition of $f_{A\cup B}$ will be controlled by infinitely many strategies, with Strategy i ensuring that the range of $F_{A\cup B}$ is correct on the interval $[c_i, c_{i+1}]$.

Strategy i will claim the interval $(\delta_i, \delta_{i+1}]$ upon which to work, where $\delta_0 = 0$ and $\delta_{i+1} = \delta_i + \epsilon_i(\epsilon_i + 1) + 1$ with $\epsilon_i = \max\{\alpha_i, \beta_i\}$. Note that the assumption on γ implies $\delta_i < \omega^\gamma$ for all i. Strategy i works by placing each element of A at least ϵ_i apart, thus leaving sufficient room for elements of B.

Strategy i: At stage $s+1$, the ith strategy defines $f_{A\cup B}(\xi, s+1)$ for every ordinal $\xi \in (\delta_i, \delta_{i+1}]$ as follows:

(1) If $\xi = \delta_{i+1}$, define $f_{A\cup B}(\delta_{i+1}, s+1) = c_{i+1,s+1}$, where $c_{i+1,s+1} = \min\{f_A(\alpha_{i+1}, s+1), f_B(\beta_{i+1}, s+1)\}$ is the current guess for the value of c_{i+1}.

(2) If $\xi = \delta_i + \epsilon_i(\alpha + 1)$ for some α and $f_A(\alpha, s+1) \in (c_{i,s+1}, c_{i+1,s+1})$, define $f_{A\cup B}(\xi, s+1) = f_A(\alpha, s+1)$.

(3) If $f_B(\beta, s+1) \in (c_{i,s+1}, c_{i+1,s+1})$, let α be least such that $f_A(\alpha, s+1) \geq f_B(\beta, s+1)$. Note that $\alpha \leq \alpha_{i+1}$. Define $f_{A\cup B}(\delta_i + \epsilon_i(\alpha) + \beta + 1, s+1) = f_B(\beta, s+1)$.

(4) If $f_{A\cup B}(\xi, s) = 0$ and $f_{A\cup B}(\xi, s+1)$ has not been defined by one of the previous cases, let $f_{A\cup B}(\xi, s+1) = 0$.

(5) If $f_{A\cup B}(\xi, s) \neq 0$, but $f_{A\cup B}(\xi, s+1)$ has not been defined by one of the previous cases, let $t < s$ be the last stage at which $f_{A\cup B}(\xi, t) \neq 0$ was defined by one of the previous cases. Note that this definition was by Case 2 or Case 3.
   (a) If $f_{A\cup B}(\xi, t)$ was defined by Case 2, and $f_A(\alpha, s+1)$ is too small (i.e., if $f_A(\alpha, s+1) \leq c_{i,s+1}$), define $f_{A\cup B}(\xi, s+1) = c_{i,s+1}$.
   (b) If $f_{A\cup B}(\xi, t)$ was defined by Case 2, and $f_A(\alpha, s+1)$ is too big (i.e., $f_A(\alpha, s+1) \geq c_{i+1,s+1}$), define $f_{A\cup B}(\xi, s+1) = c_{i+1,s+1}$.
   (c) If $f_{A\cup B}(\xi, t)$ was defined by Case 3 (so $\xi = \delta_i + \epsilon_i(\alpha) + \beta + 1$) and $f_B(\beta, s+1)$ is too big (i.e., $f_B(\beta, s+1) \geq c_{i+1,s+1}$ or $f_B(\beta, s+1) > f_A(\alpha, s+1)$), then define $f_{A\cup B}(\xi, s+1) = f_{A\cup B}(\delta_i + \epsilon_i(\alpha+1), s+1)$.
   (d) If $f_{A\cup B}(\xi, t)$ was defined by Case 3 (so $\xi = \delta_i + \epsilon_i(\alpha) + \beta + 1$) and $f_B(\beta, s+1)$ is too small ($f_B(\beta, s+1) \leq c_{i,s+1}$ or $f_B(\beta, s+1) \leq f_A(\alpha', s+1)$ for some $\alpha' < \alpha$), choose the greatest $\alpha' < \alpha$ such that $f_{A\cup B}(\xi, s) \leq f_A(\alpha', s+1)$ and $c_{i,s+1} < f_A(\alpha', s+1)$, and define $f_{A\cup B}(\xi, s+1) = f_A(\alpha', s+1)$. If there is no such α, define $f_{A\cup B}(\xi, s+1) = c_{i,s+1}$.

Construction: At stage $s+1$, each Strategy i for $i \leq s$ acts as described above to define $f_{A\cup B}(\xi, s+1)$ for all $\xi \in (\delta_i, \delta_{i+1}]$. Define $f_{A\cup B}(\xi, s+1) = 0$ for all ordinals $\xi > \delta_{s+1}$.

Verification: We argue that $f_{A\cup B}$ is a support increasing limitwise monotonic approximation on ω^γ for $A \cup B$. The construction assures that the approximation $f_{A\cup B}$ satisfies $f_{A\cup B}(\xi, s) \leq f_{A\cup B}(\xi, s+1)$ for all ξ and s and that, if the limit function exists everywhere, it is support increasing.

In order to show $F_{A\cup B}(\xi)$ exists for all $\xi \in (\delta_i, \delta_{i+1}]$, we show $f_{A\cup B}(\xi, s)$ is cofinitely often defined by the same case, from which the result is immediate. If $\xi = \delta_{i+1}$, then $f(\xi, s)$ is defined by Case 1 for all s. If $\xi = \delta_i + \epsilon_i(\alpha+1)$ and $F_A(\alpha) \in (c_i, c_{i+1})$, then there exists t such that $f_A(\alpha, s) \in (c_{i,s}, c_{i+1,s})$ for all $s > t$. Thus $f_{A\cup B}(\xi, s)$ is defined by Case 2 for all $s > t$. The other cases proceed similarly.

Thus $F_{A\cup B} \in \mathbf{SILM}(\omega^\gamma)$. It remains to show its range is $A\cup B$. The containment $\mathrm{range}(F_{A\cup B}) \subseteq A \cup B$ is immediate from the construction as $f_{A\cup B}(\xi, s) \in A \cup B$

for all ξ and s. For the reverse containment, consider an arbitrary $a \in A$. If $a = c_i$ for some i, then a will be in the range of $F_{A\cup B}$ by some Case 1. Otherwise, there is some α such that $F_A(\alpha) = a$ and some i such that $a \in (c_i, c_{i+1})$. Then $f_{A\cup B}(\delta_i + \epsilon_i(\alpha+1), s)$ is eventually always defined by Case 2 to equal a, and thus $F_{A\cup B}(\delta_i + \epsilon_i(\alpha+1)) = a$. Showing an arbitrary $b \in B$ is in the range proceeds similarly. □

Some results about the class $\mathbf{SSILM}(\omega^\alpha)$ transfer immediately from results about the class $\mathbf{SSILM}(\mathbb{Q})$.

Corollary 6.2. *For every computable ordinal α, the class $\mathbf{SSILM}(\omega^\alpha)$ is not closed under unions or joins.*

Corollary 6.3. *There is a Δ^0_2 degree that is not support strictly increasing limitwise monotonic on ω^α for any computable ordinal α.*

Breaking the pattern, a weakened version of the above also holds of $\mathbf{SILM}(w^\alpha)$.

Theorem 6.4. *For any computable ordinal α, there is a Δ^0_2 degree that is not support increasing limitwise monotonic on ω^α.*

Proof. Theorem 5.4 demonstrated the existence of a Δ^0_2 degree that is not support strictly increasing limitwise monotonic on $\mathbb{Q}$. It was not important that the domain was $\mathbb{Q}$. However it was important that we were only attempting to defeat the support strictly increasing approximations, specifically in the proof of Claim 5.4.1. With support increasing limitwise monontonic functions on ω^α, we are assured a bound exists on the number of times the strategy can reach Step 6 as the y found at each Step 5 form a strictly decreasing sequence, which must necessarily be finite. Thus the same construction suffices, modifying the verification only slightly. □

In fact, a simpler construction for Theorem 6.4 suffices that uses only one strategy for $\mathcal{R}_{e,i}$.

7. Increasing η-Representations

We continue our study of support increasing (support strictly increasing) limitwise monotonic functions on $\mathbb{Q}$ with an application to linear orders.

Definition 7.1. For an infinite set $S = \{a_0 < a_1 < a_2 < \dots\}$ of natural numbers, a *weak η-representation of S* is a linear order of the form

$$\eta + a_{F(0)} + \eta + a_{F(1)} + \eta + a_{F(2)} + \eta + \dots$$

for some surjective function $F : \omega \to \omega$. If F is bijective, then the linear order is a *unique η-representation of S*; if F is the identity, then the linear order is a *strong η-representation of S*. If F is increasing (i.e., non-decreasing), then the linear order is an *increasing η-representation.*

We refer the reader to [5], [11], and [12] for background, history, and various results about these encodings (all but the last have been previously studied) and to [3] for a general reference on linear orders in computability theory. We mention that in [5], it was shown that the sets $S \subseteq \omega$ with computable weak (unique) η-representations are exactly the $\mathbf{0}'$-limitwise monotonic sets.

Since $\eta \cong \eta + 1 + \eta$, it is undesirable to allow F to take on the value 1. We therefore introduce the following convention.

Convention 7.2. *Throughout this section, we assume any limitwise monotonic function satisfies $F(x) \neq 1$ for all x. Moreover, without loss of generality, we assume that any limitwise monotonic approximation satisfies $f(x,s) \neq 1$ for all x and s.*

The following notion is related to limitwise monotonic sets and simplifies the results for increasing η-representations.

Definition 7.3. Let $\mathcal{D} = (D : \prec)$ be a computable linear order. A function $F : D \to \omega$ is $\mathbf{d}$-*limit infimum* if there is a total $\mathbf{d}$-computable function $f : D \times \omega \to \omega$ such that $F(x) = \liminf_s f(x,s)$ exists for all $x \in D$.

A set $S \subseteq \omega$ is a *(support increasing / support strictly increasing)* $\mathbf{d}$-*limit infimum set on* $\mathcal{D}$ if there is a (support increasing / support strictly increasing) $\mathbf{d}$-limit infimum function F on $\mathcal{D}$ whose range is $S \cup \{0\}$.

A degree $\mathbf{a}$ is a *(support increasing / support strictly increasing)* $\mathbf{d}$-*limit infimum degree on* $\mathcal{D}$ if there is a set $S \in \mathbf{a}$ that is a (support increasing / support strictly increasing) $\mathbf{d}$-limit infimum set on $\mathcal{D}$.

The function $f(x,s)$ is said to be a *(support increasing / support strictly increasing)* $\mathbf{d}$-*limit infimum approximation on* $\mathcal{D}$ *for* S.

Definition 7.3 is a generalization of a definition introduced independently by Harris in [5] and Kach in [8] to an arbitrary domain rather than ω. In both [5] and [8], it was shown that the $\mathbf{0}'$-limitwise monotonic sets and the limit infimum sets coincide. Extracting the uniformity present from the proof of this equivalence, we obtain the following proposition.

Proposition 7.4. *Let $\mathcal{D} = (D : \prec)$ be a computable linear order and let $\mathbf{d}$ be any degree. A set $S \subseteq \omega$ is a (support increasing / support strictly increasing) $\mathbf{d}'$-limitwise monotonic set on $\mathcal{D}$ if and only if it is a (support increasing / support strictly increasing) $\mathbf{d}$-limit infimum set on $\mathcal{D}$.*

Moreover, uniformly in an index for a $\mathbf{d}'$-computable function $f(x,t)$, there is an index for a $\mathbf{d}$-computable function $g(x,s)$ satisfying $\liminf_s g(x,s) = \lim_t f(x,t)$ for all x. Also, uniformly in an index for a $\mathbf{d}$-computable function $g(x,s)$, there is an index for a $\mathbf{d}'$-computable function $f(x,t)$ satisfying $\lim_t f(x,t) = \liminf_s g(x,s)$ for all x.

Theorem 7.5. *For sets $S \subseteq \omega$, the following are equivalent:*

(1) *There is a computable increasing η-representation of S.*
(2) *The set S is a support increasing $\mathbf{0}'$-limitwise monotonic set on $\mathbb{Q}$.*
(3) *The set S is a support increasing limit infimum set on $\mathbb{Q}$.*

By Proposition 7.4, it suffices to show (1) $\implies$ (2) and (3) $\implies$ (1). Before showing these implications, we introduce some terminology.

Definition 7.6. If $\mathcal{L} = (L : \prec)$ is a linear order and $X \subseteq L$ is finite, a *maximal block in* X is an ordered collection (with $n > 0$) of points $\langle x_{i_0}, \dots, x_{i_n} \rangle \subseteq X$ maximal with respect to the property that x_{i_j} and $x_{i_{j+1}}$ are adjacent in $\mathcal{L}$ for all $0 \leq j \leq n-1$.

Points $x, y \in L$ are *adjacent in* $\mathcal{L}$ if $\neg(\exists z)\,[x \prec z \prec y]$.

Proof of (1) $\implies$ (2). Let $\mathcal{L} = (L : \prec)$ be a computable copy of an increasing η-representation of S with $L = \{x_n : n \in \omega\}$. Working in the presence of a $\mathbf{0}'$ oracle, we will define a $\mathbf{0}'$-computable function $f : \mathbb{Q} \times \omega \to \omega$ with the intent that $f(r,s)$ will be a $\mathbf{0}'$-limitwise monotonic approximation on $\mathbb{Q}$ for S.

We note that $\mathbf{0}'$ suffices to determine whether two elements x_i and x_j are adjacent in $\mathcal{L}$, and thus whether an ordered collection $\langle x_{i_0}, \dots, x_{i_n} \rangle \subseteq X$ is a maximal block in X. The idea will be to track these adjacencies by associating a rational number r to each maximal block B_r in $X_s = \{x_0, \dots, x_s\}$ in an order preserving manner. The value of $f(r,s)$ will then be the size of the maximal block B_r in the set X_s. As the sizes of the maximal blocks in X_s are not larger than the sizes of the maximal blocks in X_{s+1}, the function $f(r,s)$ will be increasing in s.

By extending $X_s = \{x_0, \dots, x_s\}$ to $X_{s+1} = \{x_0, \dots, x_s, x_{s+1}\}$, several scenarios can occur. We may see a new maximal block of adjacencies, in which case we will add a new rational r to the support of F. We may see an existing maximal block of adjacencies B_r grow, in which case we increase the value of $f(r,s)$ appropriately. We may see two maximal blocks of adjacencies B_{r_1} and B_{r_2} merge, in which case we associate the merged maximal block with both the rationals r_1 and r_2.

Construction: At stage 0, we define $f(r,0) = 0$ for all rationals r. At stage $s+1$, we consider the set $X_{s+1} = \{x_0, \dots, x_s, x_{s+1}\}$. Several possibilities exist.

(1) If the maximal blocks in X_{s+1} are the same as the maximal blocks in X_s, we set $f(r,s+1) = f(r,s)$ for all $r \in \mathbb{Q}$.
(2) If a new maximal block appears in X_{s+1}, we choose a rational $r' \in \mathbb{Q}$ not yet in the support of F in an order preserving manner. We set $f(r,s+1) = f(r,s)$ for all $r \neq r'$ and set $f(r',s+1)$ to be the size of the new maximal block $B_{r'}$ in X_{s+1}.
(3) If a maximal block $B_{r'}$ in X_s is a subset of a maximal block in X_{s+1}, we set $f(r,s+1) = f(r,s)$ for all $r \neq r'$ and set $f(r',s+1)$ to be the size of $B_{r'}$ in X_{s+1}.
(4) If two maximal blocks B_{r_1} and B_{r_2} in X_s merge into a single maximal block (with x_{s+1}) in X_{s+1}, we associate both rationals r_1 and r_2 with this merged maximal block. We set $f(r_1,s+1)$ and $f(r_2,s+1)$ to be the size of this new merged maximal block.

Verification: We verify that $f(r,s)$ is a support increasing $\mathbf{0}'$-limitwise monotonic approximation on $\mathbb{Q}$ for S. From the construction we have that $f(r,s)$ is computable in $\mathbf{0}'$ and that $f(r,s) \leq f(r,s+1)$ for all r and s. As the maximal blocks in $\mathcal{L}$ are associated with rationals in an order-preserving manner, the function $F(r)$ will be support increasing. As $\mathcal{L}$ was an increasing η-representation, it will have no infinite maximal blocks, and thus F is total. Moreover F will enumerate S as each maximal block will be assigned to a rational r. □

We finish proving Theorem 7.5 by demonstrating (3) $\Longrightarrow$ (1). As its proof mirrors the associated result for weak η-representations (see [5]), we leave it to a sketch.

Proof of (3) $\Longrightarrow$ (1) *(Sketch).* Let $f(r,s)$ be a support increasing limit infimum approximation on $\mathbb{Q}$ for S whose support is cofinal in $\mathbb{Q}$, which we may assume without loss of generality by Corollary 4.9. We construct a computable copy of an increasing η-representation of S in ω many stages s using $f(r,s)$.

The idea will be to build at the rational r at stage s the *suborder* $\eta + f(r,s) + \eta$. The *suborder* will be composed of three *segments*: a *left dense segment*, a *center discrete segment*, and a *right dense segment*.

If $f(r,s+1) > f(r,s)$, then an appropriate number (namely $f(r,s+1) - f(r,s)$) of points are added at the right end of the center discrete segment already built for the rational r. If $f(r,s+1) < f(r,s)$, then the appropriate number of extra points (namely $f(r,s) - f(r,s+1)$) at the right end of the discrete segment get permanently associated with the right dense segment. Regardless of the relative values of $f(r,s+1)$ and $f(r,s)$, the left dense segment and right dense segment are built towards a copy of η.

As the left dense segment and right dense segment are built towards η at cofinitely many stages, in the limit they will have order type η. As only $\liminf_s f(r,s)$ many points will remain identified with the center discrete segment for infinitely many stages, in the limit the center discrete segment will have order type $F(r)$.

We finish by noting that the isomorphism type of $\mathcal{L}$ is an increasing η-representation for S. As $f(r,s)$ was a support increasing limit infimum approximation on $\mathbb{Q}$ for S, the support of F has order type ω. Excepting the rationals in the support of F, every rational will have the suborder $\eta + 0 + \eta \cong \eta$ built for it. At a rational r in the support of F, the suborder $\eta + F(r) + \eta$ is built for it. As F was assumed to enumerate S in increasing order on its support, the isomorphism type of $\mathcal{L}$ will be an increasing η-representation of S, noting that $\eta + \eta \cong \eta$ and $\eta \cdot \eta \cong \eta$. □

Theorem 7.5 then follows. Unfortunately, we leave open the characterization of the sets $S \subseteq \omega$ possessing a computable strong η-representation. We do make several observations about the applicability of support strictly increasing $\mathbf{0}'$-limitwise monotonic approximations on $\mathbb{Q}$ to strong η-representations, however.

Noting that if the function f in the proof (3) $\implies$ (1) of Theorem 7.5 is support strictly increasing, then the resulting computable linear order is a strong η-representation of S, we obtain the following.

Proposition 7.7. *A set $S \subseteq \omega$ has a computable strong η-representation if $S \in$ $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$.*

It might be hoped that the converse of Proposition 7.7 could be proved by appropriately modifying the proof (1) $\implies$ (2) of Theorem 7.5. For example, when two blocks merge in the proof of (1) $\implies$ (2), we can associate the merged block with the rational previously associated with the left block, and slide the associations of all the blocks to the right of this block one to the left to fill in the hole. This leaves an abandoned rational on the far right, but (as S was assumed to be infinite) another block can be found which can be associated with it in an order-preserving manner. Although it is not difficult to see that every block will eventually settle down to a fixed rational, it is possible that a rational won't have a fixed block settle down to it. On such a rational, $\lim_s f(r,s) = \infty$. As a consequence of the following proposition, any attempt to fix this must rely on the uniqueness of each block size.

Proposition 7.8. *There is a set $S \subseteq \omega$ with a computable increasing η-representation but no computable strong η-representation.*[1]

Proof. Relativizing Theorem 5.3 to $\mathbf{0}'$, any set of the form $S \oplus \omega$ suffices, where S is a set in any Δ^0_3 degree not having a computable strong η-representation. The

[1]Andrey Frolov and Maxim Zubkov have independently announced this result (see [4]).

existence of such degrees can be found in [5] or follows from relativizing Theorem 5.4 to $\mathbf{0}'$ with slight modifications. □

8. Order-Computable Sets

In addition to the connection between limitwise monotonic sets and η-representations, there is a connection between limitwise monotonic sets and order-computable sets (see [6]).

Definition 8.1. A set $A \subseteq \omega$ is *order-computable* if there is a computable copy of the structure $(\omega :<, A)$ in the language of linear orders with an additional unary predicate.

Denote by **OC** the class of order-computable sets.

We demonstrate that the support increasing limitwise monotonic sets on a well-order do not contain all the order-computable sets, yet the support strictly increasing limitwise monotonic sets on $\mathbb{Q}$ contain all the order-computable sets.

Proposition 8.2. *For any computable ordinal α, there is a set $A \in$ **OC** with $A \notin$ **SILM**(ω^α).*

Proof. We build an order-computable set A that diagonalizes against all support increasing limitwise monotonic approximations on ω^α. We fix a computable enumeration $\{\varphi_i\}_{i\in\omega}$ of total computable functions with domain $\omega^\alpha \times \omega$ as in Proposition 4.1.

We describe the general strategy to defeat an individual approximation function φ_i, i.e., to assure that φ_i is not a support increasing limitwise monotonic approximation on ω^α for A.

Strategy i: The general strategy to ensure that A is not the range of Φ_i is as follows:

(1) Choose a large witness x.
(2) Wait for φ_i to match A up to x.
(3) Insert x into A by adding a new element to the underlying order at the appropriate location and declaring that the predicate holds on it. Reset Strategy j for all $j > i$.
(4) Wait for x to enter the range of φ_i.
(5) Remove x from A and insert $x + 1$ into A by adding a new element to the underlying order at the appropriate place and declaring that the predicate fails on it. Reset Strategy j for all $j > i$.
(6) Wait for φ_i to match A by increasing the column with value x to $x + 1$.
(7) Put x into A and reset Strategy j for all $j > i$, and return to Step 4,

Construction: At stage 0, we $A_s = \emptyset$. At stage $s + 1$, we start working to satisfy Strategy s as described.

Verification: It suffices to argue that each strategy spends cofinitely many stages waiting. For if every strategy spends cofinitely many stages waiting, each strategy successfully diagonalizes against Φ_i. Consequently, the set A is order-computable.

Assume each Strategy j for $j < i$ spends cofinitely many stages waiting. Then Strategy i is reset at most finitely often. Therefore Strategy i can only reach Step 4 finitely often as ω^α is well-ordered, and each new occurrence of x is to the left of the previous one (see Proposition 4.1). It follows that Strategy i also spends cofinitely many stages waiting.

By construction, the set A is order-computable. □

Proposition 8.3. *The containment* $\mathbf{OC} \subsetneq \mathbf{SSILM}(\mathbb{Q})$ *holds and is proper.*

Proof. The containment follows from the fact that a support strictly increasing limitwise monotonic approximation on $\mathbb{Q}$ can be defined for any order-computable set by putting a new element in the range of $\mathbb{Q}$ whenever the order-computable predicate holds on a new element. The approximation value is the number of predecessors on which the order-computable predicate holds.

The containment being proper follows from the existence of computably enumerable sets that are not order-computable (see [6]). Yet every computably enumerable set is easily seen to be support strictly increasing limitwise monotonic on $\mathbb{Q}$. □

As a consequence, results on order-computable sets (see [6]) can yield results about limitwise monotonic functions and η-representations.

Corollary 8.4. *Every ω-c.e. degree is support strictly increasing limitwise monotonic on $\mathbb{Q}$.*

Corollary 8.5. *Every ω-c.e. degree relative to $\mathbf{0}'$ contains a set with a computable strong η-representation.*

9. Open Questions

Many natural questions about support increasing (support strictly increasing) limitwise monotonic sets and degrees remain. Several of these stem from results in this paper.

Question 9.1. Is it the case that $\mathbf{SILM}(\mathbb{Q}) = \cup_{\alpha<\omega_1^{\mathrm{CK}}}\mathbf{SILM}(\omega^\alpha)$ or $\mathbf{SSILM}(\mathbb{Q}) = \cup_{\alpha<\omega_1^{\mathrm{CK}}}\mathbf{SSILM}(\omega^\alpha)$?

Question 9.2. Are the elements of $\mathbf{SSILM}(\mathbb{Q})$ exactly the sets with computable strong η-representations?

Question 9.3. Is there an ordinal α such that the class $\mathbf{SILM}(\omega^\alpha)$ is not closed under unions or joins?

Other questions were not addressed in this paper.

Question 9.4. Are the classes $\mathbf{SILM}(\mathbb{Q})$, $\mathbf{SSILM}(\mathbb{Q})$, $\mathbf{SILM}(\omega^\alpha)$, or $\mathbf{SSILM}(\omega^\alpha)$ closed under intersection?

Question 9.5. Are the $\mathbf{SILM}(\omega^\alpha)$ degrees the same as the $\mathbf{SILM}(\omega^{\alpha+1})$ degrees?

Question 9.6. Are there computable linear orders $\mathcal{D}_1$ and $\mathcal{D}_2$ such that neither $\mathbf{SILM}(\mathcal{D}_1) \not\subseteq \mathbf{SILM}(\mathcal{D}_2)$ nor $\mathbf{SILM}(\mathcal{D}_2) \not\subseteq \mathbf{SILM}(\mathcal{D}_1)$? Such that neither $\mathbf{SSILM}(\mathcal{D}_1) \not\subseteq \mathbf{SSILM}(\mathcal{D}_2)$ nor $\mathbf{SSILM}(\mathcal{D}_2) \not\subseteq \mathbf{SSILM}(\mathcal{D}_1)$?

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