Intrinsically Complete Relations on Linear Orders

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Definition

For α a finite ordinal, $\Delta^0_{\alpha+1} = \Delta^0_1(0^{(\alpha)})$.
For α an infinite ordinal, $\Delta^0_\alpha = \Delta^0_1(0^{(\alpha)})$.

Definition

For a linear order L, the *condensation of L* is the quotient order obtained by identifying elements which have only finitely many elements between them.
The α-*condensation of L* is the αth iterate of the condensation operation.

Note that if L is computable, the α-condensation is $\Delta^0_{2\alpha+1}$.

Definition

For a linear order L, let $S_\alpha(L)$ be the set of pairs (a, b) such that $([a], [b])$ form a successivity in the α-condensation of L.

Definition

For a linear order L, let $I_\alpha(L)$ be the set of pairs (a, b) such that $[a] \neq [b]$ in the α-condensation of L.

If L is computable, $I_\alpha(L)$ is $\Delta^0_{2\alpha+1}$, and $S_\alpha(L)$ is $\Delta^0_{2\alpha+2}$.

Theorem (Downey-Moses)

There exists a computable linear order A such that $S_0(A)$ is intrinsically Δ^0_2-Turing complete, i.e. for any computable linear order $B \cong A$, $S_0(B) \geq_T 0'$.

Since $S_0(B)$ is Δ^0_2, we can replace $S_0(B) \geq_T 0'$ with $S_0(B) \equiv_T 0'$.

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Question

Are there similar results for Δ^0_α?

Theorem (Turetsky)

For any computable ordinal α, there exists a computable linear order A on which the relation I_α is intrinsically $\Delta^0_{2\alpha+1}$-Turing complete and the relation S_α is intrinsically $\Delta^0_{2\alpha+2}$-Turing complete.

The observation that the α-condensation of L is $\Delta^0_{2\alpha+1}$ has a converse:

Theorem (Ash)

If α is a computable ordinal and L is a linear order with a $\Delta^0_{2\alpha+1}$ presentation, then $\omega^\alpha \cdot L$ possesses a computable presentation.

Note that $S_0(L)$ in some sense equals $S_\alpha(\omega^\alpha \cdot L)$.

So we might relativize the Downey-Moses result, then use the Ash result to obtain a computable linear order.

The Downey-Moses result relativized to $\mathbf{0}^{(\beta)}$:

Theorem

There exists a $\mathbf{0}^{(\beta)}$-computable linear order A_β, such that for any $\mathbf{0}^{(\beta)}$-computable linear order $B \cong A_\beta$, $S_0(B) \oplus 0^{(\beta)} \in \mathbf{0}^{(\beta+1)}$.

Letting $\beta = 2\alpha + 1$ and combining with Ash:

Theorem

For any computable ordinal α, there exists a computable linear order A such that for any computable linear order $B \cong A$, $S_\alpha(B) \oplus 0^{(2\alpha+1)} \in \mathbf{0}^{(\mathbf{2\alpha+2})}$.

(For finite α, replace $2\alpha + 1$ and $2\alpha + 2$ with 2α and $2\alpha + 1$.)

Theorem

For any computable ordinal α, there exists a computable linear order A such that for any computable linear order $B \cong A$, $S_\alpha(B) \oplus 0^{(2\alpha+1)} \in \mathbf{0^{(2\alpha+2)}}$.

The $\oplus 0^{(2\alpha+1)}$ is an artifact of the relativization. Proving the result directly gets:

Theorem

For any computable ordinal α, there exists a computable linear order A on which the relation S_α is intrinsically $\Delta^0_{2\alpha+2}$-Turing complete.

We want relations R_α which will be intrinsically $\Delta^0_{2\alpha+1}$-Turing complete.

Consider the base case of $\alpha = 1$. Then we're looking for Δ^0_3-relations on a linear order. A natural choice is "infinite".

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Consider the base case of $\alpha = 1$. Then we're looking for Δ^0_3-relations on a linear order. A natural choice is "infinite".

Note that "[*a*] and [*b*] are infinitely far apart in the α-condensation" is equal to "*a* and *b* are not identified in the $(\alpha + 1)$-condensation".

Definition

For a linear order L, let $I_\alpha(L)$ be the set of pairs (a, b) such that $[a] \neq [b]$ in the α-condensation of L.

Parallel to the Downey-Moses result, we have:

Theorem (Turetsky)

There exists a computable linear order A such that for any computable linear order $B \cong A$, $I_1(B)$ is Turing complete for Δ^0_3.

Parallel to the Downey-Moses result, we have:

Theorem (Turetsky)

There exists a computable linear order A such that for any computable linear order $B \cong A$, $I_1(B)$ is Turing complete for Δ^0_3.

Relativizing and combining with Ash, we again have:

Theorem

For any computable successor ordinal α, there exists a computable linear order A such that for any computable linear order $B \cong A$, $I_\alpha(B) \oplus 0^{(2\alpha-1)} \in \mathbf{0}^{(2\alpha+1)}$.

Theorem

For any computable successor ordinal α, there exists a computable linear order A such that for any computable linear order $B \cong A$, $I_\alpha(B) \oplus 0^{(2\alpha-1)} \in \mathbf{0}^{(2\alpha+1)}$.

Still two problems with this:

- Has an extra $\oplus 0^{(2\alpha-1)}$.
- Only works for successor ordinals.

Both artifacts of the relativization. Proving directly gets:

Theorem

For any computable ordinal α, there exists a computable linear order A on which the relation I_α is intrinsically $\Delta^0_{2\alpha+1}$-Turing complete.

Want to construct an A with $S_0(A)$ intrinsically Δ_2^0-Turing complete.

Fix X a Σ_1^0-complete set.

For computable linear order L_i, we want to ensure that if $L_i \cong A$, then $S_0(L_i)$ computes X.

For each L_i, we will build an A_i to handle L_i.

Note

For the corresponding I_1 result, use Y a Σ_2^0-complete set in place of X.

Definition

η is the order-type of $\mathbb{Q}$.

$A = A_0 + 3 + A_1 + 4 + A_2 + \dots$

A_i will have type $(\eta + 2 + \eta) \cdot \tau$ for some τ.

τ will have type $n + \omega^*$, $\omega + n$ or $\omega + \omega^*$.

Note that $n + \omega^*$, $\omega + n$ and $\omega + \omega$ each have the property that no element has infinitely many points to its left and to its right.

Note

For the corresponding I_1 result, A_i will have type $(\eta + \zeta + \eta) \cdot \tau$. ($\zeta$ the order-type of $\mathbb{Z}$).

We will build Γ_i a Turing functional and endeavor to make $\Gamma_i^{S_0(L_i)} = X$.

Begin building A_i so τ has type ω.

Enumerate a computation into Γ_i for our guess of $X(n)$ with use our guess of the first $n+1$ adjacencies.

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Begin building A_i so τ has type ω.

Enumerate a computation into Γ_i for our guess of $X(n)$ with use our guess of the first $n+1$ adjacencies.

When our guess of $X(n)$ changes, we need to change our computation. So we need to change the use of the old computation, but we don't directly control L_i. So instead we attack the isomorphism.

Consider the arrangement of the first $n+1$ adjacencies in L_i. Kill all later adjacencies in A_i, then put the new $n+1$ adjacency in the same relative position as the first in L_i.

Example ($X(8)$ Changes)

L is:

$$\cdots \ell_5 \cdots \ell_3 \cdots \ell_8 \cdots \ell_0 \cdots \ell_1 \cdots \ell_2 \cdots \ell_4 \cdots \ell_7 \cdots \ell_6 \cdots$$

Our behavior on A:

$$\cdots a_0 \cdots a_1 \cdots a_5 \cdots a_6 \cdots a_7 \cdots a_8 \cdots a_4 \cdots a_3 \cdots a_2 \cdots$$

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If the use of the old computation never changes, then ℓ_0 cannot correspond to a_j for any $j < n$.

If this occurs for infinitely many n, then there can be no isomorphism between L_i and A_i.

If this occurs for only finitely many n, then Γ_i correctly computes X except for finitely many elements.

The actual reduction cannot be found uniformly.

Putting it all together, we have the following:

Theorem (Turetsky)

For any computable ordinal α, there exists a computable linear order A on which the relation I_α is intrinsically $\Delta^0_{2\alpha+1}$-Turing complete and the relation S_α is intrinsically $\Delta^0_{2\alpha+2}$-Turing complete.

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Question

For a computable limit ordinal α, is there a relation R_α which is intrinsically Δ^0_α-Turing complete?

Rod Downey and Michael Moses
Recursive Linear Orders with Incomplete Successivities
Transactions of the American Mathematical Society, Vol. 326, No. 2 (Aug., 1991), pp.132-142

Christopher Ash
A Construction for Recursive Linear Orderings
The Journal of Symbolic Logic, Vol. 56, No. 2 (Jun., 1991), pp. 673-683

Christopher Ash and Julia Knight
Computable Structures and the Hyperarithmetical Hierarchy

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Untitled
In preparation