

Dimension Level Sets in the Plane

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Basics

Disconnectivity Results

Connectivity Results

Main Result

Questions

Vague idea:

Effective dimension of Z = Amount that Z can be
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Less vague idea:

Effective dimension of Z =
$$\frac{\# \text{ of bits needed to compute } Z \text{ with } r \text{ bits of precision}}{r}$$

For $\sigma \in 2^{<\omega}$, what is meant by “number of bits needed to compute σ ”?

Need a notion of complexity of σ .

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Need a notion of complexity of σ .

Several different options:

$C(\sigma)$ — Plain Kolmogorov complexity.

$K(\sigma)$ — Prefix-free Kolmogorov complexity.

$Km(\sigma)$ — Monotone complexity.

$KM(\sigma)$ — Solomonoff complexity?

Notions differ by at most $2 \log |\sigma|$.

Now generalize those notions to \mathbb{R}^n :

Fix U a universal Turing machine. Fix a numbering $\delta^n : 2^{<\omega} \rightarrow \mathbb{Q}^n$.

Let $U^n = \delta^n \circ U$.

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$$C_r(z) := \min\{|\tau| : U^n(\tau) \in B(z, 2^{-r})\}.$$

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Definition

For $z \in \mathbb{R}^n$, define

$$\dim(z) := \liminf_{r \rightarrow \infty} \frac{C_r(z)}{r}.$$

Examples:

- Let $\alpha \in \mathbb{R}^1$ be a computable real. Then α has dimension 0.
- Let $\beta \in \mathbb{R}^1$ be a random real. Then β has dimension 1.
- $\beta \oplus \emptyset$ (β with a 0 inserted every other bit) has dimension $1/2$.
- Similarly, every rational less than 1 can be the dimension of a real.
- With a little more work, every real less than 1 can be the dimension of a real.

Facts:

- For $z \in \mathbb{R}^n$, $0 \leq \dim(z) \leq n$.
- Every real satisfying the above occurs as the dimension of densely many points.
- Almost every point has dimension n .
- If $(x, y) \in \mathbb{R}^2$, $\dim((x, y)) \geq \dim(x)$.
- If $q \in \mathbb{Q}$, $\dim((x, q)) = \dim(x)$.
- If $(x, y) \in \mathbb{R}^2$, $\dim((x, y)) = 2 \dim(x \oplus y)$.

Lemma

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is sufficiently computable and Lipschitz on a neighborhood of $z \in \mathbb{R}^n$, then $\dim(f(z)) \leq \dim(z)$.

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Proof.

Let 2^s be the Lipschitz constant.

Then $f(B(z, 2^{-r})) \subseteq B(f(z), 2^{-r+s})$.

So $C_{r-s}(f(z)) \leq C_r(z) + O(1)$.

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{C_r(f(z))}{r} &= \liminf_{r \rightarrow \infty} \frac{C_{r-s}(f(z))}{r-s} \\ &\leq \liminf_{r \rightarrow \infty} \frac{C_r(z) + O(1)}{r-s} = \dim(z). \quad \square \end{aligned}$$

Now we restrict our attention to \mathbb{R}^2 .

Definition

Let $S \subseteq [0, 2]$. Define

$$\text{DIM}^S := \{z \in \mathbb{R}^2 : \dim(z) \in S\}.$$

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Question (Lutz & Weihrauch)

What are the connectivity properties of DIM^S for various S ?

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Question (Lutz & Weihrauch)

What are the connectivity properties of DIM^S for various S ?

Possibilities:

Path-Connected \Rightarrow Connected

Totally-Disconnected \Rightarrow Punctiform \Rightarrow Pathwise Totally-Disconnected

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Theorem (Lutz & Weihrauch)

For $S \subseteq [0, 1)$ or $S \subseteq (1, 2]$, DIM^S is totally disconnected.

Proof on next page.

In fact, a later result will show:

Theorem (Turetsky)

For S not containing 1, DIM^S is punctiform (the only compact connected subsets are singletons).

Proof.

For $q \in \mathbb{Q}$, consider the line $\ell : x_i = q$.

For $S \subseteq (1, 2]$, $\ell \cap \text{DIM}^S = \emptyset$.

So given two points in DIM^S , they differ in at least one of their coordinates. So we can find an ℓ which separates them.

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For $S \subseteq (1, 2]$, $\ell \cap \text{DIM}^S = \emptyset$.

So given two points in DIM^S , they differ in at least one of their coordinates. So we can find an ℓ which separates them.

For $r \in \mathbb{R}$ with $\dim(r) = 1$, consider the line $\ell : x_i = r$.

For $S \subseteq [0, 1)$, $\ell \cap \text{DIM}^S = \emptyset$.

So given two points in DIM^S , they differ in at least one of their coordinates. So we can find an ℓ which separates them. \square

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Theorem (Lutz & Weihrauch)

For $S \supseteq [0, 1]$ or $S \supseteq [1, 2]$, DIM^S is path-connected.

Follows from next lemma.

In fact, a later result will show:

Theorem (Turetsky)

$DIM^{\{1\}}$ is connected.

Lemma (Lutz & Weihrauch)

If x, y and z are distinct collinear points, then

$$\dim^x(y) \leq \dim(z) + 1.$$

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Proof.

$y = t(z - x)$ for some t .

This is Lipschitz in a neighborhood of x, z and t .

Taking x as an oracle, this is an x -computable function f with $f(z, t) = y$. So

$$\dim^x(y) \leq \dim^x(z, t) \leq \dim(z) + 1. \quad \square$$

Lemma (Lutz & Weihrauch)

If x, y and z are distinct collinear points, then

$$\dim^x(y) \leq \dim(z) + 1.$$

Proof of Lutz & Weihrauch path-connected result:

If x and z are distinct rational points, every point on the line segment between them has dimension at most 1.

Given two points, choose sequences of rational points converging to them.

Connect consecutive points with line segments. This gives us a path between the two points. Every point on the interior of the path has dimension at most 1.

Lemma (Lutz & Weihrauch)

If x, y and z are distinct collinear points, then

$$\dim^x(y) \leq \dim(z) + 1.$$

Proof of Lutz & Weihrauch path-connected result:

If y has dimension 2 relative to x , every point on the line segment between them has dimension at least 1.

Given two points, choose a sequence starting with the first and converging to the second, with every point in the sequence having dimension 2 relative to the previous point.

Connect consecutive points with line segments. This gives us a path. Every point in the interior of the path has dimension at least 1.

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Every non-singleton, closed, connected subset of the plane contains a point of dimension 1.

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Corollary

For S not containing 1, DIM^S is punctiform (the only compact connected subsets are singletons).

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Corollary

For S not containing 1, DIM^S is punctiform (the only compact connected subsets are singletons).

Proof.

A compact connected subset of DIM^S is closed in \mathbb{R}^2 , so if it's not a singleton, it contains a point of dimension 1. \square

Theorem (Turetsky)

Every non-singleton, closed, connected subset of the plane contains a point of dimension 1.

Corollary

$DIM^{\{1\}}$ is connected.

Proof.

If $U \cup V$ is a separation of $DIM^{\{1\}}$, then $X = \mathbb{R}^2 - U - V$ is a closed set.

Connected components are closed, so by the above theorem, X is totally disconnected.

A closed, totally disconnected subset of \mathbb{R}^n is dimension 0, and a dimension 0 set cannot separate the plane. $\Rightarrow \Leftarrow$ □

Theorem (Turetsky)

Every non-singleton, closed, connected subset of the plane contains a point of dimension 1.

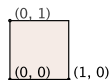
Sketch:

- Without loss of generality, work within the unit square.
- Find a new notion of complexity.
- Prove a connectedness lemma.
- Find a way to force complexity up.
- Find a way to force complexity down.

First, terminology:

A *dyadic square* is one that can be obtained by repeatedly quartering the unit square:

Width 1 (unit square)



Width $1/2$



Width $1/4$



Solomonoff complexity for \mathbb{R}^2 :

Definition

Let M be an optimal enumerable semi-measure on \mathbb{R}^2 (dominates every enumerable semi-measure up to a multiplicative constant). For a point $z \in \mathbb{R}^2$ and $r \in \omega$, let $S_r(z)$ be the dyadic square containing z of width 2^{-r} . Define

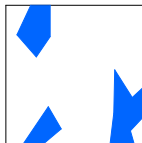
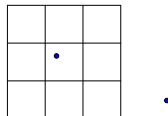
$$KM_r(z) := -\log M(S_r(z)).$$

Lemma

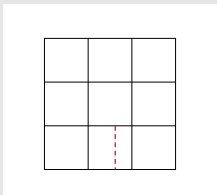
If a closed, connected set X contains a point in the middle square and a point outside all the squares,

then for at least one of the squares, the intersection of X with that square contains at least half the width or half the height of the square.

Further, for at least one of the squares, the intersection of X with that square contains an interval of width or an interval of height.

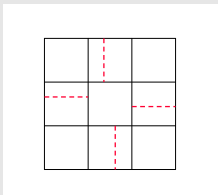


Proof.



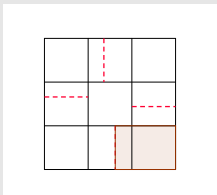
Suppose X contains the entire width of the bottom square. Then we're done. Otherwise, there is some vertical line in the bottom square missed by X .

Proof.



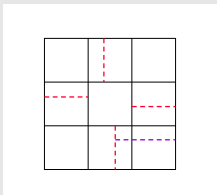
Similarly, there are lines missed by X in the top, middle-left and middle-right squares.

Proof.



Now consider the region to the right of the bottom line. If X has the entire height, by additivity and the Baire Category theorem we are done.

Proof.



If not, there is some horizontal line in the region missed by X .

Forcing complexity up:

Divide the current square into 2^{2k} many squares.

Each has width 2^{-k} times the width of the current square.

Half the width or height means the set intersects at least 2^{k-1} of them.

At least one has measure no more than 2^{1-k} of the measure of the current square.

So at least one increases complexity by at least $k - 1$.

Forcing complexity down:

Containing an interval of width or interval of height means the set in this square contains a point with a rational coordinate.

This point has dimension at most 1. If dimension 1, we're done.

Otherwise, the dimension is less than 1, so some dyadic square around it of width 2^{-r} has complexity less than r .

Stop at the first such box to get complexity within 1 bit of r .

Construction:

Start with a small dyadic square intersecting the set in a large amount of height or width.

Choose a smaller dyadic square with high complexity (forcing the complexity up).

Switch to an adjacent dyadic square of the same size with a large amount of the set, via the lemma.

Repeat.

If complexity ever rises above $r + 1$, force the complexity down.

Shifting to an adjacent square of width 2^{-r} changes complexity by at most $2 \log r$.

So we potentially lose $2 \log r + 1$ bits of complexity at every stage.

So complexity is at least $r - 2n \log r - n$ at stage n .

So make r increase quadratically in n .

Then $\lim_{r \rightarrow \infty} \frac{r - 2n \log r - n}{r} = 1$.

Infinitely often, the complexity is at most $r + 1$.

So the dimension of our point is 1. \square

Open Question

Is $DIM^{\{1\}}$ path-connected?

If not, precisely how much of $[0, 2]$ is needed for path-connectedness?

Lines, or polynomial curves in general, will not suffice for $DIM^{\{1\}}$.

Open Question

Is $DIM^{[0,1) \cup (1,2]}$ connected?



Jack Lutz and Klaus Weihrauch

Connectivity Properties of Dimension Level Sets

Mathematical Logic Quarterly, Vol. 54 (2008), pp. 483-491



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Dimension One Points in the Plane

In preparation