First-Order Linear Differential Equations

Thursday, 20 September 2007

STANDARD FORM

Given two continuous functions $P(x)$ and $Q(x)$ of x defined on an interval $I \subseteq \mathbb{R}$ the corresponding **first-order linear differential equation** in **standard form** is the equation

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,.$$

When $Q(x) \equiv 0$ the differential equation is **homogeneous**.

Otherwise the differential equation is called **inhomogenous**.

When $P(x) \equiv 0$ the solution is the antiderivative $y(x) = \int Q(x)\,dx\,.$

SOLUTION OF THE HOMOGENEOUS EQUATION:

This equation is **separable** since it takes the form

$$\frac{dy}{dx} = -P(x)\,y\,.$$

For $y \neq 0$ we find the **one-parameter family** of solutions

$$\begin{aligned}
\int \frac{dy}{y} &= -\int P(x)\,dx \\
\ln|y| &= -\int P(x)\,dx \\
|y| &= e^{-\int P(x)\,dx} \\
y(x) &= C\,e^{-\int P(x)\,dx}
\end{aligned}$$

SOLUTION OF THE INHOMOGENEOUS EQUATION:

We try a solution of the form

$$y(x) \;=\; u(x)\, e^{-\int P(x)\, dx} \;=\; u(x)\, \left(e^{\int P(x)\, dx}\right)^{-1} .$$

Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx}\, e^{-\int P(x)\, dx} + u(x)\, \frac{d}{dx}\left(e^{-\int P(x)\, dx}\right) \\
&= \frac{du}{dx}\, e^{-\int P(x)\, dx} + u(x)\, e^{-\int P(x)\, dx}\, \frac{d}{dx}\left(-\int P(x)\, dx\right) \\
&= \frac{du}{dx}\, e^{-\int P(x)\, dx} + u(x)\, e^{-\int P(x)\, dx}\, (-P(x)) \\
&= \frac{du}{dx}\, e^{-\int P(x)\, dx} - P(x)\, y
\end{aligned}$$

$$
\begin{aligned}
\frac{dy}{dx} + P(x)\,y &= \frac{du}{dx}\,e^{-\int P(x)\,dx} \\
\frac{dy}{dx} + P(x)\,y &= Q(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{du}{dx}\,e^{-\int P(x)\,dx} &= Q(x) \\
\frac{du}{dx} &= Q(x)\,e^{\int P(x)\,dx} \\
u(x) &= \int Q(x)\,e^{\int P(x)\,dx}\,dx
\end{aligned}
$$

SUMMARY:

The general solution of the **first-order linear differential equation** takes the form

$$y(x) \;=\; e^{-\int P(x)\,dx} \int Q(x)\, e^{\int P(x)\,dx}\, dx\,.$$

The integration constant hidden in the notation $\int Q(x)\, e^{\int P(x)\,dx}\, dx$ produces a one-parameter family of solutions.

The analogous constant in $\int P(x)\, dx$ cancels from the formula.

Any one of the one-parameter family of functions $v(x) = e^{\int P(x)\,dx}$ appearing in the above solution formula is called an **integrating factor** for the differential equation.

STEPS TO FIND THE SOLUTION:

1. Bring the equation into the **normal form** $\frac{dy}{dx} + P(x)y = Q(x)$ and **identify** $P(x),\ Q(x)$
2. Compute the **antiderivative** $\int P(x)\,dx$ and form the **integrating factor** $e^{\int P(x)\,dx}$
3. Transformed equation: $\frac{d}{dx}\left(e^{\int P(x)\,dx}\ y\right) = e^{\int P(x)\,dx}\,Q(x)$
4. Antiderive this equation: $e^{\int P(x)\,dx}\ y = \int Q(x)e^{\int P(x)\,dx}\,dx$
5. Solve for y: $\quad y(x) = e^{-\int P(x)\,dx}\ \int Q(x)e^{\int P(x)\,dx}\,dx$

Simple Resistance-Inductance Electrical Circuits

Consider the RL-circuit shown in FIGURE 9.5 on page 654. The current i flowing in this circuit at time t satisfies the following differential equation involving the **inductance** $L \neq 0$ in `Henry`, the **resistance** R in `Ohm`, the **voltage** V in `Volt`:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}.$$

In the simplest case, when L, R, V are **constant** in time, this linear differential equation has the integrating factor

$$e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}.$$

The general solution is therefore given by

$$\begin{aligned} i(t) &= e^{-\int \frac{R}{L}\,dt} \int \frac{V}{L}\, e^{\int \frac{R}{L}\,dt}\, dt \\ &= e^{-\frac{R}{L}t} \left(\int \frac{V}{L}\; e^{\frac{R}{L}t}\, dt \right) \\ &= e^{-\frac{R}{L}t} \left(\frac{V}{L} \frac{L}{R}\; e^{\frac{R}{L}t} \; + \; C \right) \\ &= \frac{V}{R} + C\, e^{-\frac{R}{L}t} \end{aligned}$$

If $i(0) = 0$ the constant C equals $-\frac{V}{R}$ and the solution is a superposition of the **steady-state** and the **transient** solutions:

$$i(t) \;=\; \frac{V}{R} - \frac{V}{R}\, e^{-\frac{R}{L}t}\,.$$