Problem 1

(5 points)  Find the standard-form equation for the hyperbola which has its foci at \(F_{\pm}(\pm 2, 0)\) and whose asymptotes are \(y = \pm \frac{1}{\sqrt{3}}x\).

The calculations

\[
\begin{align*}
\frac{b}{a} &= \frac{1}{\sqrt{3}} \\
a &= \sqrt{3}b \\
c^2 &= a^2 + b^2 = 2^2 = 4 \\
c^2 &= 3b^2 + b^2 = 4b^2 = 4 \\
b^2 &= 1 \\
b &= 1 \\
a &= \sqrt{3}b = \sqrt{3}
\end{align*}
\]

yield the standard-form equation

\[
x^2 - \frac{y^2}{1} = 1.
\]

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(5 points)  Find the standard-form equation for the ellipse which has a focus at \(F(4, 0)\) and whose eccentricity is \(e = \frac{2}{3}\).

The calculations

\[
\begin{align*}
c &= 4 \\
e &= \frac{c}{a} \\
\frac{2}{3} &= \frac{4}{a} \\
a &= \frac{3}{2} = 6 \\
a^2 &= b^2 + c^2 \\
36 &= b^2 + 16 \\
b^2 &= 20
\end{align*}
\]

yield the standard-form equation

\[
\frac{x^2}{36} + \frac{y^2}{20} = 1.
\]
(5 points) Replace the Cartesian equation \( x^2 + (y-2)^2 = 4 \) by the equivalent polar equation and simplify.

The calculations

\[
\begin{align*}
  x^2 + (y-2)^2 &= 4 \\
  x^2 + y^2 - 4y + 4 &= 4 \\
  x^2 + y^2 &= 4y \\
  r^2 &= 4r \sin \theta \\
  r &= 4 \sin \theta \quad \text{or} \quad r = 0
\end{align*}
\]

result in the equivalent polar equation

\[ r = 4 \sin \theta. \]

Note: Since the point with \( r = 0 \) is part of the solution set for this equation (when \( \theta = 0 \)), no point is lost by working only with this polar equation.

(6 points) Find the slope of the curve \( r = -1 + \sin \theta \) at the point on it where \( \theta = 0 \). The function involved in the given polar equation, \( f(\theta) = -1 + \sin \theta \) has the value \( f(0) = -1 \). Its derivative \( f'(\theta) = \frac{df}{d\theta} = \cos \theta \) has the value \( f'(0) = \cos 0 = 1 \).

\[
\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}
\]

\[
\left( \frac{dy}{dx} \right)_{\theta=0} = \frac{f'(0) \sin 0 + f(0) \cos 0}{f'(0) \cos 0 - f(0) \sin 0} = \frac{f(0) \cos 0}{f'(0) \cos 0} = \frac{(-1)(1)}{(1)(1)} = -1
\]
Find the shared area inside the two curves $r = 2 \cos \theta$ and $r = 2 \sin \theta$ where $0 \leq \theta \leq \pi$.

The shared boundary points are found to be $P(r, \theta) = P(0, \theta)$ and $Q(r, \theta) = Q(2, \frac{\pi}{4})$. The line $\theta = \frac{\pi}{4}$ is an axis of reflection symmetry for the region in question. Hence the area can be computed by

\[
\text{Area} = 2 \int_{0}^{\pi/4} \frac{1}{2} (2 \sin(\theta))^2 \ d\theta
\]

\[
= \int_{0}^{\pi/4} 4 \sin^2(\theta) \ d\theta
\]

\[
= \int_{0}^{\pi/4} 2(1 - \cos(2\theta)) \ d\theta
\]

\[
= \frac{2\pi}{4} - 2 \sin(\frac{2\pi}{4}) \bigg|_{0}^{\pi/4}
\]

\[
= \frac{\pi}{2} - \sin(\frac{\pi}{2})
\]

\[
= \frac{\pi}{2} - 1.
\]
Consider the sequence $a_n = \left( \frac{3n+1}{3n-1} \right)^n$.

(10 points) Determine if this sequence converges or diverges. In case this sequence diverges provide a verification of its divergence. In case this sequence converges find its limit.

\[
\ln a_n = n \ln \left( \frac{3n+1}{3n-1} \right) = n \left( \ln(3n+1) - \ln(3n-1) \right) = \frac{\ln(3n+1) - \ln(3n-1)}{n^{-1}}.
\]

The limit of this logarithm when $n \to \infty$ is of the form $\frac{0}{0}$. L'Hôpital’s rule yields the computation

\[
\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{\ln(3n+1) - \ln(3n-1)}{n^{-1}} = \lim_{n \to \infty} \frac{3 \left( \frac{1}{3n+1} - \frac{1}{3n-1} \right)}{-n^{-2}} = -3 \lim_{n \to \infty} \frac{n^2((3n-1) - (3n+1))}{(3n+1)(3n-1)} = -3 \lim_{n \to \infty} \frac{-2n^2}{9n^2 - 1} = \frac{2}{3}.
\]

It follows from the continuity of the exponential function that the sequence $a_n$ converges to the limit

\[
L = \lim_{n \to \infty} a_n = e^{\lim_{n \to \infty} \ln a_n} = e^{2/3} = \sqrt[3]{e^2}.
\]
(5 points) Express the number $-2.5\overline{417} = -2.5417417417417\ldots$ as a ratio of two integers.

\[
2.5 = \frac{25}{10}
\]

\[
0.0\overline{417} = \frac{417}{10^4} \sum_{n=0}^{\infty} 10^{-3n}
\]

\[
= \frac{417}{10^4} \frac{1}{1 - 10^{-3}} = \frac{417}{10^4} \frac{10^3 - 1}{10^3 - 1} = \frac{417}{(10)(999)} = \frac{417}{9990}
\]

\[
2.5\overline{417} = \frac{25}{10} + \frac{417}{9990} = \frac{(25)(999) + 417}{9990} = \frac{25392}{9990} = \frac{4232}{1665}
\]

\[
-2.5\overline{417} = \frac{-4232}{1665}.
\]

Consider the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + (\ln n)^2)} \).

(4 points) Verify that the hypotheses of the Integral Test are satisfied by this series.

The series terms are \( f(n) = (n(1 + (\ln n)^2))^{-1} \) for the function \( f(x) = (x(1 + (\ln x)^2))^{-1} \). For \( x > 0 \) this function is positive and continuous, since \( 1 + (\ln x)^2 \geq 1 \).

The functions \( x, \ln x, (\ln x)^2, 1 + (\ln x)^2, x(1 + (\ln x)^2) \) are all increasing. Therefore, as the reciprocal of a positive, increasing function, \( f(x) \) is decreasing.

The hypotheses of the integral test are satisfied for the given series and the function \( f(x) \).

(3 points) Evaluate the integral arising from the integral test applied to the above series in order to determine its convergence or divergence.

Using the substitution \( u = \ln x \), \( du = \frac{dx}{x} \) one calculates

\[
\int_{1}^{\infty} (x(1 + (\ln x)^2))^{-1} \, dx = \int_{0}^{\infty} (1 + u^2)^{-1} \, du = \arctan(u) \bigg|_{0}^{\infty} = \frac{\pi}{2}.
\]

According to the Integral Test the convergence of this integral is equivalent to the convergence of the given series.
Consider the series \[ \sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}}. \]

\begin{enumerate}
\item \textbf{(5 points)} Determine if this series converges or diverges by applying the Limit Comparison Test.

Replacing the term \( n+1 \) by \( n \) in the numerator of the series yields the comparison terms \[ \frac{n}{n^2 \sqrt{n}} = n^{1-2-1/2} = n^{-3/2}, \]
which are the terms of the convergent \( p \)-series with \( p = \frac{3}{2} \).
Since \[ \lim_{n \to \infty} \frac{n+1}{n^2 \sqrt{n}} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1 \notin \{0, \infty\} \]
the Limit Comparison Test implies the convergence of the series.
\end{enumerate}

Consider the series \[ \sum_{n=1}^{\infty} \frac{(\ln \sqrt{n})^n}{n^{n/2}}. \]

\begin{enumerate}
\item \textbf{(6 points)} Determine if this series converges or diverges by using an appropriate test.

The terms of the series are positive for \( n > 1 \).
In order to apply the Root Test one calculates (using L'Hôpital's rule for a limit of the form \( \frac{\infty}{\infty} \))

\[ \sqrt[n]{\frac{(\ln \sqrt{n})^n}{n^{n/2}}} = \frac{\ln \sqrt{n}}{n^{1/2}} = \frac{\ln n^{1/2}}{n^{1/2}} \]
\[ \rho = \lim_{n \to \infty} \frac{\ln n^{1/2}}{n^{1/2}} = \lim_{n \to \infty} n^{-1/2} \frac{d}{dn} n^{1/2} \]
\[ = \lim_{n \to \infty} n^{-1/2} \]
\[ = 0. \]

Since \( \rho = 0 \), the Root Test guarantees the convergence of this series.
Consider the series \[ \sum_{n=1}^{\infty} (-1)^n \left( \sqrt{n+1} - \sqrt{n} \right) \].

(5 points) Determine if this series is convergent.

Hint: What is the value of \( (\sqrt{n+1} - \sqrt{n}) (\sqrt{n+1} + \sqrt{n}) \)?

\[
(\sqrt{n+1} - \sqrt{n}) (\sqrt{n+1} + \sqrt{n}) = (\sqrt{n+1})^2 - (\sqrt{n})^2 = (n+1) - n = 1.
\]

Writing \( u_n = (\sqrt{n+1} + \sqrt{n})^{-1} \) the terms of the given series are of the form \( a_n = (-1)^n u_n \), with \( u_n > 0 \).

This shows that the given series is alternating.

Since the sequences \( \sqrt{n+1} \), \( \sqrt{n} \), \( \sqrt{n+1} + \sqrt{n} \) are all positive and increasing, the reciprocal of the last one, \( u_n \), is decreasing.

Since \( \lim_{n \to \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \) the Alternating Series Convergence Test applies and assures the convergence of the alternating series.

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(4 points) Determine if this series is absolutely convergent.

The series of absolute values is

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \sum_{n=1}^{\infty} u_n .
\]

The \( p \)-series for \( p = 1/2 \) is divergent.

Comparing the given series to this \( p \)-series the Limit Comparison Test

\[
\lim_{n \to \infty} \frac{1/\sqrt{n}}{1/(\sqrt{n+1} + \sqrt{n})} = \lim_{n \to \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n}}
\]

\[
= \lim_{n \to \infty} \left( \sqrt{1 + \frac{1}{n}} + 1 \right)
\]

\[
= 2 \notin \{0, \infty\}
\]

shows that the absolute value series is divergent.

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(2 points) Determine if this series is conditionally convergent.

The given alternating series is convergent but not absolutely convergent.

By definition, the given series is therefore conditionally convergent.