Conic Sections and Polar Coordinates

10.6 Graphing

10.7 Areas and Lengths

10.8 Conic Sections

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Polar to Rectangular Coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta. \tag{1} \]

If \( r = 0 \) and \( \theta \in \mathbb{R} \) the described point \( P(x, y) \) is the origin \((0, 0)\). If any other point \( P(x, y) \) is described by polar coordinates \((r, \theta)\) or \((r', \theta')\) then these coordinates are related by

\[ r \cos \theta = r' \cos \theta' \quad \text{and} \quad r \sin \theta = r' \sin \theta' \]

or equivalently

\[ r' = r \quad \text{and} \quad \theta' = \theta + (2n)\pi, \quad n \in \mathbb{Z} \]

or

\[ r' = -r \quad \text{and} \quad \theta' = \theta + (2n + 1)\pi, \quad n \in \mathbb{Z}. \]
Rectangular to Polar Coordinates:
For a point $P(x, y)$ different from the origin a polar coordinate description is given by

$$
r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x} \text{ if } x \neq 0 \text{ or } \theta = \cot^{-1} \frac{x}{y} \text{ if } y \neq 0.
$$

(2)
Symmetry: Reflection in $x$-axis

$$(x, y) \mapsto (x, -y)$$

$$(r \cos(\theta), r \sin(\theta)) \mapsto (r \cos(\theta), -r \sin(\theta))$$

$$= (r \cos(-\theta), r \sin(-\theta))$$

$$= (-r \cos(\pi - \theta), -r \sin(\pi - \theta))$$

$$(r, \theta) \mapsto (r, -\theta)$$
or

$$(r, \theta) \mapsto (-r, \pi - \theta).$$
Symmetry: Reflection in $y$-axis

$$(x, y) \mapsto (-x, y)$$

$$(r \cos(\theta), r \sin(\theta)) \mapsto (-r \cos(\theta), r \sin(\theta))$$

$$= (r \cos(\pi - \theta), r \sin(\pi - \theta))$$

$$= (-r \cos(-\theta), -r \sin(-\theta))$$

$$(r, \theta) \mapsto (r, \pi - \theta)$$

or

$$(r, \theta) \mapsto (-r, -\theta).$$
Symmetry: Reflection in origin:

\[
(x, y) \mapsto (-x, -y) \\
\iff \\
(r \cos(\theta), r \sin(\theta)) \mapsto (-r \cos(\theta), -r \sin(\theta)) \\
= (r \cos(\theta + \pi), r \sin(\theta + \pi)) \\
\iff \\
(r, \theta) \mapsto (-r, \theta) \\
or \\
(r, \theta) \mapsto (r, \theta + \pi). 
\]
Slope of the Curve $r = f(\theta)$:

Given a function $r = f(\theta)$ the equations

$$x = f(\theta) \cos \theta \quad \text{and} \quad y = f(\theta) \sin \theta$$

provide a parameterized description of a curve in the $(x, y)$-plane.

Assuming $f'(\theta) \cos \theta - f(\theta) \sin \theta \neq 0$, the slope of the tangent to this parameterized curve can be computed by

$$\frac{dy}{dx} = \frac{dy}{d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.$$ 

When $f(\theta) = 0$ the slope equals $\frac{dy}{dx} = \tan \theta$, while $f'(\theta) = 0$ implies $\frac{dy}{dx} = -\cot \theta$, so that in that case the tangent is perpendicular to the ray from the origin to the point $P(x, y)$. 
Area in the Plane:
The area described by the conditions
\( \alpha \leq \theta \leq \beta, \ 0 \leq r \leq f(\theta) \) is given by

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta .
\]  
(4)

The area described by the conditions
\( \alpha \leq \theta \leq \beta, \ 0 \leq f_1(\theta) \leq r \leq f_2(\theta) \) is given by

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} \left( (f_2(\theta))^2 - (f_1(\theta))^2 \right) \, d\theta .
\]  
(5)
Length of a Polar Curve:

Assuming that \( r = f(\theta) \) is continuously differentiable for \( \alpha \leq \theta \leq \beta \) and that the point \( P_{\text{polar}}(r, \theta) \) traces the graph exactly once, the length of the curve is given as follows

\[
\begin{align*}
    dx &= \left( f'(\theta) \cos \theta - f(\theta) \sin \theta \right) d\theta \\
    dy &= \left( f'(\theta) \sin \theta + f(\theta) \cos \theta \right) d\theta \\
    ds^2 = dx^2 + dy^2 &= \left( f'^2(\theta) + f^2(\theta) \right) d\theta^2
\end{align*}
\]

so that

\[
L = \int ds = \int_{\alpha}^{\beta} \sqrt{f'^2(\theta) + f^2(\theta)} \, d\theta .
\]  \hspace{1cm} (6)
Area of Surface of Revolution of a Polar Curve

Assuming that \( r = f(\theta) \) is continuously differentiable for \( \alpha \leq \theta \leq \beta \) and that the point \( P_{\text{polar}}(r, \theta) \) traces the graph exactly once, the areas of the surfaces generated by revolving the curve around the \( x \)- and \( y \)-axes is given as follows

\[
x-axis: \quad S = \int_{\alpha}^{\beta} 2\pi f(\theta) \sin \theta \sqrt{f'(\theta)^2 + f^2(\theta)} \, d\theta \quad (7)
\]

\[
y-axis: \quad S = \int_{\alpha}^{\beta} 2\pi f(\theta) \cos \theta \sqrt{f'(\theta)^2 + f^2(\theta)} \, d\theta \quad (8)
\]
Polar Equations for Lines:
If the perpendicular to a line $L$ from the origin meets the line at the point $P_{polar}(r_0, \theta_0)$, where $r_0 > 0$ then the general point $P_{polar}(r, \theta)$ of the line $L$ satisfies the equation
\[ r \cos(\theta - \theta_0) = r_0. \] (9)

Polar Equations for Circles:
The general point $P_{polar}(r, \theta)$ of the circle with center $P_{polar}(r_0, \theta_0)$ and radius $a > 0$ satisfies the equation
\[ a^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0). \] (10)
Ellipses, Parabolas, and Hyperbolas:
Using the eccentricity $e > 0$ in the focus-directrix definition of the conic sections, where the focus $F$ is assumed at the origin and the directrix $D$ is described by $x = k$ for some $k > 0$, one finds the polar equations

\[
P F = e P D
\]

\[
r = e(k - x)
\]

\[
r = e(k - r \cos \theta)
\]

\[
r(1 + e \cos \theta) = ek \quad \text{so that, finally}
\]

\[
r = \frac{ek}{1 + e \cos \theta}.
\]
Standard Ellipse:
An ellipse (not a circle) has an eccentricity $e$ with $0 < e < 1$. According to Figure 10.19 a translation in the $x$-direction by $-c$ verifies that $k = \frac{a}{e} - c$ so that $ek = a - ec = \left(1 - e^2\right)a$ and

$$r = \frac{(1 - e^2)a}{1 + e \cos \theta}.$$  \hfill (12)
**Standard Parabola:**
A parabola has eccentricity \( e = 1 \).
It is seen that \( k = 2p > 0 \) so that the equation for the parabola opening up to the left and with axis equal to the \( x \)-axis is

\[
r = \frac{2p}{1 + \cos \theta}.
\]

(13)
Standard Hyperbola:
A hyperbola has an eccentricity $e$ with $e > 1$.
According to Figure 10.20 a translation in the $x$-direction by $c$ verifies that $k = c - \frac{a}{e}$ so that $ek = ec - a = \left( e^2 - 1 \right) a$ and
\[
r = \frac{\left( e^2 - 1 \right) a}{1 + e \cos \theta}, \quad \text{with } |\theta| < \cos^{-1}\left( \frac{1}{e} \right). \tag{14}
\]