Infinite Sequences and Series

Section 11.1

Example 11.1.1

$$
\begin{aligned}
a_n &= \left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n+\frac{1}{2}} \\
&= \left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n+\frac{1}{2}}\cdot\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) \\
&= \left(\sqrt{n+1}^2-\sqrt{n}^2\right)\cdot\left(\frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n+1}+\sqrt{n}}\right) \\
&= \frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n+1}+\sqrt{n}} \\
&= \frac{\frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n}}}{\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}}} \\
&= \frac{\sqrt{1+\frac{1}{2n}}}{\sqrt{1+\frac{1}{n}}+\sqrt{1}}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{2n}}}{\sqrt{1+\frac{1}{n}}+\sqrt{1}} \\
&= \frac{\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{2n}}\right)}{\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1}\right)} \\
&= \frac{\left(\sqrt{1+\lim_{n\to\infty}(\frac{1}{2n})}\right)}{\left(\sqrt{1+\lim_{n\to\infty}(\frac{1}{n})}+\sqrt{1}\right)} \\
&= \frac{\left(\sqrt{1+0}\right)}{\left(\sqrt{1+0}+\sqrt{1}\right)} \\
&= \frac{(1)}{(1+1)} \\
&= \frac{1}{2}
\end{aligned}
$$

Example 11.1.2

$$1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

$$a_1 = 1, \quad a_{n+1} = \sqrt{2\,a_n}\,, \quad for\ n \geq 1$$

Clearly $a_1 \leq 2$. Assuming that $a_k \leq 2$ it follows that $a_{k+1} = \sqrt{2a_k} \leq \sqrt{2 \cdot 2} = 2$. By induction it follows that for all $n \in \mathbb{N}$, $a_n \leq 2$.

Since the function $x \mapsto \sqrt{x}$ is increasing we see that

$$\begin{aligned} a_n &\leq 2 \text{ implies} \\ \sqrt{a_n} &\leq \sqrt{2}\,, \text{ so that} \\ a_n &= \sqrt{a_n}\sqrt{a_n} \\ &\leq \sqrt{2}\,\sqrt{a_n} \\ &= \sqrt{2a_n} \\ &= a_{n+1}\,, \text{ or finally} \\ a_n &\leq a_{n+1}\,. \end{aligned}$$

By Theorem 6
$\lim_{n\to\infty} a_n = L$ exists with $(1 \le L \le 2)$.

Hence

$$\begin{aligned}
L &= \lim_{n\to\infty} a_{n+1} \\
&= \lim_{n\to\infty} \sqrt{2\,a_n} \\
&= \sqrt{2 \lim_{n\to\infty} a_n} \\
&= \sqrt{2\,L}, \text{ so that} \\
L &= \sqrt{2\,L} \\
L^2 &= 2\,L \\
0 &= 2\,L - L^2 \\
0 &= L(2-L) \\
L &\in \{0, 2\}
\end{aligned}$$

Since $1 \le L \le 2$ it follows that

$$\boxed{\lim_{n\to\infty} a_n = L = 2}\,.$$

Example 11.1.3

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}\,, \quad for\ n \geq 1$$

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

$\{a_n\}$ is increasing since $x \mapsto \sqrt{x}$ is increasing.

Clearly $a_1 < 3$. Assuming that $\sqrt{2} < a_k < 3$ it follows that
$\sqrt{2} < a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+3} = \sqrt{5} < 3$.
Induction then shows that the sequence $\{a_n\}$ is bounded with $\sqrt{2} < a_n < 3$

By Theorem 6 $\lim_{n\to\infty} a_n = L$ exists with $\sqrt{2} \leq L \leq 3$.
Hence

$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \sqrt{2 + a_n} \\ &= \sqrt{2 + \lim_{n\to\infty} a_n} \\ &= \sqrt{2 + L} \\ L^2 &= 2 + L \\ 0 &= L^2 - L - 2 \\ L &\in \{-1, 2\} \end{aligned}$$

Since $\sqrt{2} \leq L \leq 3$ it follows that

$$\boxed{\lim_{n\to\infty} a_n = L = 2}\,.$$

Example 11.1.4 Given:

f, a continuous function

a, a constant

define the sequence $\{a_n\}$ by

$a_1 = a,\ a_{n+1} = f(a_n)$

If $\lim_{n\to\infty} a_n = L$, then

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(L)$

$\boxed{L = f(L)}$

<u>Illustration</u>:

$f(x) = \cos x,\ a = 1$

$a_1 = 1,\ a_{n+1} = \cos(a_n)$

$\{1, \cos(1), \cos(\cos(1)), \cos(\cos(\cos(1))), \ldots\}$

$\boxed{L = \cos(L) \text{ has the solution } L \approx 0.7390851332}$

Example 11.1.5 A continued fraction.

Define $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{1+a_n}$, $n \geq 1$
Then $a_2 = 1 + \frac{1}{1+a_1} = 1 + \frac{1}{1+1} = \frac{3}{2}$

$$\begin{aligned} a_{n+2} &= 1 + \frac{1}{1 + a_{n+1}} \\ &= 1 + \frac{1}{1 + 1 + \frac{1}{1+a_n}} \\ &= 1 + \frac{1 + a_n}{3 + 2a_n} \\ &= \frac{4 + 3a_n}{3 + 2a_n} \end{aligned}$$

Since $a_n \geq 1 \;\Rightarrow \frac{1}{1+a_n} \leq \frac{1}{2} \;\Rightarrow a_{n+1} \leq \frac{3}{2}$
so that the sequence is bounded.

Since the derivative $\frac{d\left[\frac{4+3a}{3+2a}\right]}{da} = 3 + 2a^{-2} > 0$ is positive, it follows that

$$\boxed{a_{n+2} > a_n}\,.$$

Hence any progression in steps of length **two** is increasing.

By Theorem 6 $\lim_{n\to\infty} a_{2n} = L_{even} > 0$ exists.

By Theorem 6 $\lim_{n\to\infty} a_{2n+1} = L_{odd} > 0$ also exists.

Both limits satisfy the equation

$$L = \frac{4+3L}{3+2L},\ L > 0\,.$$

Since this equation only has the solutions $\pm\sqrt{2}$ it follows that

$$\boxed{L = L_{even} = L_{odd} = \lim_{n\to\infty} a_n = \sqrt{2}}\,.$$