Infinite Sequences and Series

Section 11.1
Example 11.1.1

\[ a_n = (\sqrt{n+1} - \sqrt{n}) \sqrt{n + \frac{1}{2}} \]

\[ = (\sqrt{n+1} - \sqrt{n}) \sqrt{n + \frac{1}{2}} \cdot \left( \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) \]

\[ = (\sqrt{n+1^2} - \sqrt{n^2}) \cdot \left( \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n+1 + \sqrt{n}}} \right) \]

\[ = \frac{\sqrt{n + \frac{1}{2}}}{\sqrt{n+1 + \sqrt{n}}} \]

\[ = \frac{\sqrt{n + \frac{1}{n}}}{\sqrt{n+1} + \sqrt{n}} \]

\[ = \frac{\sqrt{1 + \frac{1}{2n}}}{\sqrt{1 + \frac{1}{n} + \sqrt{1}}} \]
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\sqrt{1 + \frac{1}{2n}}}{\sqrt{1 + \frac{1}{n} + \sqrt{1}}}

= \lim_{n \to \infty} \left( \sqrt{1 + \frac{1}{n}} + \sqrt{1} \right)

= \left( \sqrt{1 + \lim_{n \to \infty} \left( \frac{1}{2n} \right)} \right)

= \frac{(\sqrt{1 + 0})}{(\sqrt{1 + 0} + \sqrt{1})}

= \frac{(1)}{(1 + 1)}

= \frac{1}{2}$$
Example 11.1.2

\[1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots\]

\[a_1 = 1, \quad a_{n+1} = \sqrt{2} a_n, \quad \text{for } n \geq 1\]

Clearly \(a_1 \leq 2\). Assuming that \(a_k \leq 2\) it follows that \(a_{k+1} = \sqrt{2} a_k \leq \sqrt{2} \cdot 2 = 2\). By induction it follows that for all \(n \in \mathbb{N}\), \(a_n \leq 2\).

Since the function \(x \mapsto \sqrt{x}\) is increasing we see that

\[
\begin{align*}
a_n & \leq 2 \quad \text{implies} \\
\sqrt{a_n} & \leq \sqrt{2}, \quad \text{so that} \\
a_n & = \sqrt{a_n a_n} \\
& \leq \sqrt{2} \sqrt{a_n} \\
& = \sqrt{2a_n} \\
& = a_{n+1}, \quad \text{or finally} \\
a_n & \leq a_{n+1}.
\end{align*}
\]
By Theorem 6
lim_{n \to \infty} a_n = L \exists \text{ with } (1 \leq L \leq 2).

Hence

\[
L = \lim_{n \to \infty} \sqrt{2} a_n \\
= \sqrt{2} \lim_{n \to \infty} a_n \\
= \sqrt{2L}, \text{ so that}
\]

\[
L = \sqrt{2L} \\
L^2 = 2L \\
0 = 2L - L^2 \\
0 = L(2 - L) \\
L \in \{0, 2\}
\]

Since \(1 \leq L \leq 2\) it follows that

\[
\lim_{n \to \infty} a_n = L = 2.
\]
Example 11.1.3

\[ a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad \text{for } n \geq 1 \]

\[ \sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots \]

\{a_n\} is increasing since \( x \mapsto \sqrt{x} \) is increasing.

Clearly \( a_1 < 3 \). Assuming that \( \sqrt{2} < a_k < 3 \) it follows that

\[ \sqrt{2} < a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 3} = \sqrt{5} < 3. \]

Induction then shows that the sequence \( \{a_n\} \) is bounded with \( \sqrt{2} < a_n < 3 \).
By Theorem 6 $\lim_{n \to \infty} a_n = L$ exists with $\sqrt{2} \leq L \leq 3$.

Hence

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n \to \infty} a_n} = \sqrt{2 + L}
\]

\[
L^2 = 2 + L
\]

\[
0 = L^2 - L - 2
\]

\[
L \in \{-1, 2\}
\]

Since $\sqrt{2} \leq L \leq 3$ it follows that

\[
\lim_{n \to \infty} a_n = L = 2.
\]
**Example 11.1.4** Given:

- \( f \), a continuous function
- \( a \), a constant

Define the sequence \( \{a_n\} \) by

\[
a_1 = a, \quad a_{n+1} = f(a_n)
\]

**If** \( \lim_{n \to \infty} a_n = L \), then

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} f(a_n) = f(L)
\]

**Illustration:**

\( f(x) = \cos x, \ a = 1 \)

\[
a_1 = 1, \quad a_{n+1} = \cos(a_n)
\]

\( \{1, \cos(1), \cos(\cos(1)), \cos(\cos(\cos(1))), \ldots\} \)

\( L = \cos(L) \) has the solution \( L \approx 0.7390851332 \)
Example 11.1.5 A continued fraction.

Define $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{1+a_n}$, $n \geq 1$

Then $a_2 = 1 + \frac{1}{1+a_1} = 1 + \frac{1}{1+1} = \frac{3}{2}$

\[
a_{n+2} = 1 + \frac{1}{1 + a_{n+1}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + a_n}}} = 1 + \frac{1 + a_n}{3 + 2a_n} = \frac{4 + 3a_n}{3 + 2a_n}
\]

Since $a_n \geq 1 \Rightarrow \frac{1}{1+a_n} \leq \frac{1}{2} \Rightarrow a_{n+1} \leq \frac{3}{2}$

so that the sequence is bounded.
Since the derivative \( \frac{d}{da} \left[ \frac{4 + 3a}{3 + 2a} \right] = 3 + 2a^{-2} > 0 \) is positive, it follows that

\[
a_{n+2} > a_n.
\]

Hence any progression in steps of length \textbf{two} is increasing.

By Theorem 6 \( \lim_{n \to \infty} a_{2n} = L_{\text{even}} > 0 \) exists.

By Theorem 6 \( \lim_{n \to \infty} a_{2n+1} = L_{\text{odd}} > 0 \) also exists.

Both limits satisfy the equation

\[
L = \frac{4 + 3L}{3 + 2L}, \quad L > 0.
\]

Since this equation only has the solutions \( \pm \sqrt{2} \) it follows that

\[
L = L_{\text{even}} = L_{\text{odd}} = \lim_{n \to \infty} a_n = \sqrt{2}.
\]