APPLICATIONS OF POWER SERIES

CHAPTER 11
SECTION 10

Suppose for $|x-a| < R$, where $R > 0$:

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = \lim_{n\to\infty} T_n(x)$$

$T_n(x) \doteq \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ and the remainders

$$R_n(x) \doteq f(x) - T_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Some methods for estimating the magnitude of the remainder:

Graphing: Graph $|R_n(x)|$

Alternating Series: $|R_n(x)| \leq \frac{|f^{(n+1)}(a)|}{(n+1)!}|x-a|^{n+1}$

Taylor's Formula: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Example 11.10.1

Einstein's Special Theory of Relativity (1905). Velocity dependent total energy $E = m\,c^2$ of an object with rest-mass m_0, velocity v:

$$
\begin{aligned}
m\,c^2 &= m_0\,c^2 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = m_0\,c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\
&= m_0\,c^2 \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{v^2}{c^2}\right)^n \\
&= m_0\,c^2 \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n\,n!} \frac{v^{2n}}{c^{2n}} \\
&= m_0\,c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \dots\right] \\
&= m_0\,c^2 + \frac{1}{2} m_0\,v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \dots
\end{aligned}
$$

First order remainder term:

$$f(x) \doteq m_0\, c^2 \frac{1}{\sqrt{1+x}}\,, \quad x = -\frac{v^2}{c^2}$$

There exists z with

$$-1 < -\frac{v^2}{c^2} < z < 0$$

such that the first order error is

$$R_1(-\frac{v^2}{c^2}) = \left(m_0\, c^2\right) \frac{3}{8}(1-z)^{-5/2} \left(-\frac{v^2}{c^2}\right)^2$$

$$= \frac{3}{8}\, m_0\, \frac{v^4}{c^2}\,(1-z)^{-5/2}$$

Example 11.10.2

$$f(x) = \ln x\,, \qquad a = 4\,,\ n = 3\,,\ 3 \le x \le 5\,.$$

$$\begin{aligned}
\ln x &= \ln\left(4 - (4 - x)\right) \\
&= \ln\left[4 \cdot \left(1 - (1 - \frac{x}{4})\right)\right] \\
&= \ln 4 + \ln\left(1 - (1 - \frac{x}{4})\right) \\
&= \ln 4 - \sum_{n=1}^{\infty} \frac{(1 - \frac{x}{4})^n}{n} \\
&= \ln 4 - \frac{(1 - \frac{x}{4})^1}{1} - \frac{(1 - \frac{x}{4})^2}{2} - \frac{(1 - \frac{x}{4})^3}{3} - \dots \\
&= \ln 4 + \frac{(x - 4)}{4} - \frac{(x - 4)^2}{32} + \frac{(x - 4)^3}{192} \mp \dots
\end{aligned}$$

Alternatively, from the definitions:

$$
\begin{aligned}
f(x) &= \boxed{\ln x} \\
f(4) &= \ln 4 \\
f'(x) &= x^{-1} \\
f'(4) &= 4^{-1} = \frac{1}{4} \\
f''(x) &= (-1)\, x^{-2} \\
f''(4) &= (-1)\, 4^{-2} = -\frac{1}{16} \\
f'''(x) &= (-1)(-2)\, x^{-3} \\
f'''(4) &= (-1)(-2)\, 4^{-3} = \frac{1}{32}
\end{aligned}
$$

. . . .

$$
\begin{aligned}
T_3(x) = \ln 4 &+ \frac{1}{1 \cdot 4}\,(x-4) \\
&- \frac{1}{2 \cdot 16}\,(x-4)^2 + \frac{1}{6 \cdot 32}\,(x-4)^3
\end{aligned}
$$

. . . .

$$
\begin{aligned}
f''''(x) &= (-1)(-2)(-3)\, x^{-4} = \boxed{-6\, x^{-4}} \\
R_3(x) &= \frac{f''''(z)}{4!}\,(x-4)^4 = \frac{-6\, z^{-4}}{4!}\,(x-4)^4 \\
1 &\geq |x-4|, \qquad z > 3 \Rightarrow z^{-4} < 3^{-4} \\
|R_3(x)| &\leq \frac{6}{3^4\, 4!} = \boxed{\frac{1}{324} \approx 0.0030864198}
\end{aligned}
$$

Example 11.10.3

Find n in Taylor's formula to estimate $e^{0.1}$ with an error less than 10^{-5}.

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{0.1} = \sum_{i=0}^{\infty} \frac{0.1^i}{i!}$$

$$= \sum_{i=0}^{n} \frac{0.1^i}{i!} + R_n(0.1)$$

$$= 1 + \frac{10^{-1}}{1} + \frac{10^{-2}}{2} + \frac{10^{-3}}{6} + \cdots + \frac{10^{-n}}{n!} + R_n(0.1)$$

. . . .

Taylor's formula:

$$\begin{aligned} R_n(0.1) &= \frac{e^z}{(n+1)!}(0.1)^{n+1} \\ &< \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} \\ &< \boxed{\frac{3^{0.1}}{(n+1)!}(0.1)^{n+1}} \\ &< 10^{-5} \end{aligned}$$

. . . .

$$R_1(0.1) < \frac{3^{0.1}}{(1+1)!}(0.1)^{1+1} \approx 0.00558062$$

$$R_2(0.1) < \frac{3^{0.1}}{(2+1)!}(0.1)^{2+1} \approx 0.000186021$$

$$R_3(0.1) < \frac{3^{0.1}}{(3+1)!}(0.1)^{3+1} \approx 4.65051\,10^{-6}$$

$$R_4(0.1) < \frac{3^{0.1}}{(4+1)!}(0.1)^{4+1} \approx 9.30103\,10^{-8}$$

$$R_5(0.1) < \frac{3^{0.1}}{(5+1)!}(0.1)^{5+1} \approx 1.55017\,10^{-9}$$

$\boxed{n = 3}$ is sufficient.

Example 11.10.4

From the Encyclopedia Britannica:

Maxwell's theory is a theory of waves in a continuous (i.e., infinitely divisible) medium. The energy of the waves is also infinitely divisible so that an indefinitely small amount can be emitted or absorbed by matter. Classical physical theories of the 19th century had predicted that in such a system the energy in equilibrium would be distributed so as to give an equal amount to each mode (frequency) of vibration. Because a continuous medium has an infinite number of modes of vibration, and the atoms (which constitute matter) have only a finite number, all the energy of the universe would be transformed into waves of high frequency. Maxwell understood this difficulty, which was later most clearly stated in the Rayleigh-Jeans law (after two English physicists, Lord Rayleigh and Sir James Hopwood Jeans) of the radiation of a blackbody.

$$f(\lambda) = 8\pi kT\lambda^{-4}$$

λ wavelength in meters

T temperature in degrees Kelvin

k Boltzmann's constant

h Planck's constant

c speed of light in vacuum

The German physicist Max Planck demonstrated that it is necessary to postulate that radiant-heat energy is emitted only in finite amounts, which are now called quanta.

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

$$\text{Consider} \quad \lambda^{-1} \to 0$$

Planck's blackbody radiation law:

$$\frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\sum_{n=1}^{\infty} \frac{(hc/(\lambda kT))^n}{n!}}$$
$$\approx \frac{8\pi hc\lambda^{-5}}{\frac{(hc/(\lambda kT))^1}{1!}}$$
$$= 8\pi kT\lambda^{-4}$$

The Rayleigh-Jeans blackbody radiation law appears as the leading term in the long wavelength expansion of the Planck law.

Example 11.10.5

CLAIM: $e = \exp(1)$ is irrational.

PROOF by contradiction.

HYPOTHESIS: e is rational:

$\exp(1) \equiv e = \frac{p}{q}$, with $p, q > 2$ positive integers.

Taylor's formula yields $0 < z < 1$:

$$e = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right) + \frac{e^z}{(q+1)!}$$

$$= s_q + \frac{e^z}{(q+1)!}, \qquad \text{with}$$

$$s_q \doteq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}$$

THEN:

$$e - s_q = \frac{e^z}{(q+1)!}$$

$$= \frac{p}{q} - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$$

Consider the real number $\mathcal{Q} \equiv q!(e - s_q)$:

$$\mathcal{Q} = q! \cdot \frac{p}{q} - \sum_{i=0}^{q} \frac{q!}{i!} \;\boxed{\text{is an INTEGER}}$$

$$\mathcal{Q} = q! \cdot \frac{e^z}{(q+1)!} = \frac{e^z}{(q+1)}, \qquad \text{so that}$$

$$0 < \mathcal{Q} = \frac{e^z}{(q+1)} < \frac{e^1}{(q+1)} < \frac{e}{(2+1)} < 1$$

Since there exists no integer $\mathcal{Q}$ which lies strictly between 0 and 1,
we arrive at a CONTRADICTION.
This forces the conclusion that the HYPOTHESIS is $\boxed{\text{untenable}}$.

CONCLUSION:

$\boxed{e \text{ is not rational, hence } \textbf{irrational}\,\textbf{.}}$