BINOMIAL SERIES

Chapter 11
SECTION 10

DEFINITION: $\boxed{(1+x)^k \doteq \exp(k \cdot \ln(1+x))}$

k real constant, $x > -1$

n^{th} derivative:

$$\left((1+x)^k\right)' = k \cdot (1+x)^{k-1}$$

$$\left((1+x)^k\right)'' = k(k-1) \cdot (1+x)^{k-2}$$

$$\left((1+x)^k\right)''' = k(k-1)(k-2) \cdot (1+x)^{k-3}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\left((1+x)^k\right)^{(n)} = k(k-1)\ldots(k-n+1) \cdot (1+x)^{k-n}$$

DEFINITION: **Binomial coefficients**:

$$\binom{k}{n} \doteq \frac{k(k-1)\ldots(k-n+1)}{n!}$$

Maclaurin series for $(1+x)^k$:

$$\sum_{n=0}^{\infty} \frac{\left((1+x)^k\right)^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Ratio test for convergence:

$$\left|\frac{\binom{k}{n+1}}{\binom{k}{n}}\right| = \frac{|k-n|}{n+1}\,|x|$$

$$\lim_{n\to\infty}\left|\frac{\binom{k}{n+1}}{\binom{k}{n}}\right| = \lim_{n\to\infty}\frac{|k-n|}{n+1}\,|x| = |x|$$

$$\therefore \quad R = 1\,.$$

It can be verified that the MacLaurin series represents the function:

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n, \quad -1 < x < 1\,.$$

This is the **BINOMIAL SERIES**.

Example:

$$
\begin{aligned}
\arcsin x &= \int_0^x \frac{1}{\sqrt{1-u^2}}\, du \\
&= \int_0^x \boxed{\Big(1 + (-u^2)\Big)^{-1/2}}\, du \\
&= \int_0^x \boxed{\sum_{n=0}^{\infty} \binom{-1/2}{n} (-u^2)^n}\, du \\
&= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \boxed{\int_0^x u^{2n}\, du} \\
&= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \boxed{\frac{1}{2n+1}\, x^{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n\,(2n+1)\,n!}\, x^{2n+1}
\end{aligned}
$$

using the so-called *double factorial*

$$
(2n-1)!! \doteq (2n-1)\cdot(2n-3)\cdots 5\cdot 3\cdot 1
$$

Initial Maclaurin - Taylor polynomials for

arcsin x :

$$T_1(x) = T_2(x) = x$$

$$T_3(x) = T_4(x) = x + \frac{x^3}{6}$$

$$T_5(x) = T_6(x) = x + \frac{x^3}{6} + \frac{3\,x^5}{40}$$

$$T_7(x) = T_8(x) = x + \frac{x^3}{6} + \frac{3\,x^5}{40} + \frac{5\,x^7}{112}$$

$$T_9(x) = T_{10}(x) = x + \frac{x^3}{6} + \frac{3\,x^5}{40} + \frac{5\,x^7}{112} + \frac{35\,x^9}{1152}$$

Example 1

$$
\begin{aligned}
\frac{x+x^2}{(1-x)^3} &= (x+x^2)\cdot \boxed{(1+(-x))^{-3}} \\
&= (x+x^2)\cdot \boxed{\sum_{n=0}^{\infty}\binom{-3}{n}(-x)^n} \\
&= (x+x^2)\cdot \sum_{n=0}^{\infty}\binom{-3}{n}(-1)^n\,x^n \\
&= (x+x^2)\cdot \sum_{n=0}^{\infty}\frac{3\cdot 4\cdot 5\ldots(n+2)}{n!}\,x^n \\
&= (x+x^2)\cdot \sum_{n=0}^{\infty}\frac{(n+2)!}{1\cdot 2\cdot n!}\,x^n \\
&= (x+x^2)\cdot \sum_{n=0}^{\infty}\frac{(n+1)\cdot(n+2)}{2}\,x^n \\
&= x\ +\sum_{n=1}^{\infty}\frac{(n+1)\cdot(n+2)}{2}\,x^{n+1} \\
&\qquad +\sum_{n=0}^{\infty}\frac{(n+1)\cdot(n+2)}{2}\,x^{n+2} \\
&= x+\sum_{n=2}^{\infty}\left[\frac{n(n+1)}{2}+\frac{(n-1)n}{2}\right]x^n \\
&= x+\sum_{n=2}^{\infty}n^2\,x^n = \boxed{\sum_{n=1}^{\infty}n^2\,x^n}
\end{aligned}
$$

$$
\frac{1/2+(1/2)^2}{(1-1/2)^3} = 6 = \sum_{n=1}^{\infty}\frac{n^2}{2^n}, \quad \text{for } x = 1/2
$$

Example 2 Period of pendulum of length L, with maximum angle θ_0 from vertical, g gravitational acceleration and $k = \sin\left(\frac{\theta_0}{2}\right)$:

$$
\begin{aligned}
& 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}(1-k^2\sin^2 x)^{-1/2}\,dx \\
&= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-k^2\sin^2 x\right)^n\,dx \\
&= 4\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n k^{2n}\boxed{\int_0^{\pi/2}\sin^{2n} x\,dx} \\
&= 4\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n k^{2n}\boxed{\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\,\frac{\pi}{2}} \\
&= 2\pi\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{2^n\,n!}\right)^2 k^{2n} \\
&= 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2 3^2}{2^2 4^2}k^4+\frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6+\ldots\right] \\
&= 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\ldots\right]
\end{aligned}
$$