Infinite Series

Section 11.2

Terms a_k and **partial sums** s_n:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \qquad (1)$$

$$\begin{aligned}
s_1 &= s_0 + a_1 = 0 + a_1 \\
s_2 &= s_1 + a_2 = a_1 + a_2 \\
s_3 &= s_2 + a_3 = a_1 + a_2 + a_3 \\
s_4 &= s_3 + a_4 = a_1 + a_2 + a_3 + a_4 \\
&\ldots \\
s_n &= s_{n-1} + a_n = \sum_{k=1}^{n} a_k
\end{aligned}$$

$$\begin{aligned}
a_1 &= s_1 - s_0 \\
a_2 &= s_2 - s_1 \\
a_3 &= s_3 - s_2 \\
a_4 &= s_4 - s_3 \\
&\ldots \\
a_n &= s_n - s_{n-1}
\end{aligned}$$

Infinite sums:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= L \in \mathbb{R} \\ &\Leftrightarrow \\ \lim_{N\to\infty} s_N &= L \in \mathbb{R}\end{aligned}$$

If $\displaystyle\sum_{n=1}^{\infty} a_n = L \in \mathbb{R}$ **then**

$$\begin{aligned}\lim_{N\to\infty} a_N &= \lim_{N\to\infty} (s_N - s_{N-1}) \\ &= \lim_{N\to\infty} s_N - \lim_{N\to\infty} s_{N-1} \\ &= L - L \\ &= 0\end{aligned}$$

Conclusion: If $\lim_{N\to\infty} a_N$ does not exist or exists but is different from 0, then the series (1) diverges.
Note, however, that $\lim_{N\to\infty} a_N = 0$ does not imply that the series (1) does converges.

Example 11.2.1

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{3\ (3\,n-2)} - \frac{1}{3\ (3\,n+1)} \right)$$
$$= \left(\frac{1}{3}\right) \sum_{n=1}^{\infty} \left(\frac{1}{(3\,n-2)} - \frac{1}{(3\,n+1)} \right)$$
$$= \left(\frac{1}{3}\right) \Bigg[\left(\frac{1}{(3\cdot 1-2)} - \frac{1}{(3\cdot 1+1)} \right)$$
$$+ \left(\frac{1}{(3\cdot 2-2)} - \frac{1}{(3\cdot 2+1)} \right)$$
$$+ \left(\frac{1}{(3\cdot 3-2)} - \frac{1}{(3\cdot 3+1)} \right) + \ldots \Bigg]$$

continuing below ...

continuing from above ...

$$
\begin{aligned}
= \quad & \left(\frac{1}{3}\right)\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)\right. \\
& \left. +\left(\frac{1}{10}-\frac{1}{13}\right)+\left(\frac{1}{13}-\frac{1}{16}\right)+\left(\frac{1}{16}-\frac{1}{19}\right)\ldots\right] \\
= & \frac{1}{3}
\end{aligned}
$$

Example 10.2.2

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2\,n} - \frac{1}{1+n} + \frac{1}{2\,(2+n)} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2\,n} - \frac{1}{2\,(1+n)} + \frac{1}{2\,(2+n)} - \frac{1}{2\,(1+n)} \right)$$

$$= \left(\frac{1}{2\cdot 1} - \frac{1}{2\,(1+1)} + \frac{1}{2\,(2+1)} - \frac{1}{2\,(1+1)} \right)$$

$$+ \left(\frac{1}{2\cdot 2} - \frac{1}{2\,(1+2)} + \frac{1}{2\,(2+2)} - \frac{1}{2\,(1+2)} \right)$$

$$+ \left(\frac{1}{2\cdot 3} - \frac{1}{2\,(1+3)} + \frac{1}{2\,(2+3)} - \frac{1}{2\,(1+3)} \right) + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Example 11.2.3

Find the values of x for which

$$\sum_{n=0}^{\infty} 2^n \sin^n x$$

converges as a geometric series?

<u>Convergence criterion</u>:

$$|2 \sin x| < 1$$
$$-\frac{1}{2} < \sin x < \frac{1}{2}$$

This happens, when (n integer)

$$-\frac{\pi}{6} + n\pi < x < \frac{\pi}{6} + n\pi$$

The sum s is then equal to

$$s = \frac{1}{1 - 2\sin x}$$

Example 11.2.4

Assume that $a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ **converges.** (2)

$$\textbf{Claim}: \quad \sum_{n=1}^{\infty} \frac{1}{a_n} \quad \textbf{diverges} \tag{3}$$

Proof.

$\lim_{n\to\infty} a_n = 0$ since (2) is convergent.

Therefore $\lim_{n\to\infty} \frac{1}{a_n}$ does not exist and

the sum in eq. (3) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

□

Example 11.2.5

$$\text{Consider } \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \tag{4}$$

Some numerical values for k and the partial sums $s_k = \sum_{n=1}^{k} \frac{n}{(n+1)!}$ are:

$$\left(1, \tfrac{1}{2}\right)$$
$$\left(2, \tfrac{5}{6}\right)$$
$$\left(3, \tfrac{23}{24}\right)$$
$$\left(4, \tfrac{119}{120}\right)$$
$$\left(5, \tfrac{719}{720}\right)$$
$$\left(6, \tfrac{5039}{5040}\right)$$
$$\left(7, \tfrac{40319}{40320}\right)$$

Hypothesis: $$\boxed{s_k = \sum_{n=1}^{k} \frac{n}{(n+1)!} = \frac{(k+1)!-1}{(k+1)!}}$$

Proof. Induction on $k \geq 1$

Induction base: $k = 1$

$s_1 = \sum_{n=1}^{1} \frac{n}{(n+1)!} = \frac{1}{(1+1)!} \doteq \frac{(1+1)!-1}{(1+1)!} = \frac{1}{2}$

Induction step: Show $s_{k+1} \doteq \frac{(k+2)!-1}{(k+2)!}$

$$\boxed{s_{k+1}} = \sum_{n=1}^{k+1} \frac{n}{(n+1)!}$$

$$= \sum_{n=1}^{k} \frac{n}{(n+1)!} + \frac{k+1}{((k+1)+1)!}$$

$$= s_k + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)\left((k+1)!-1\right)+(k+1)}{(k+2)!}$$

$$= \frac{(k+2)!-(k+2)+(k+1)}{(k+2)!}$$

$$= \boxed{\frac{(k+2)!-1}{(k+2)!}}$$

Therefore

$$\begin{aligned}
\boxed{s\ } &= \lim_{k\to\infty} s_k \\
&= \lim_{k\to\infty} \frac{(k+1)!-1}{(k+1)!} \\
&= \lim_{k\to\infty} \left(1 - \frac{1}{(k+1)!}\right) \\
&= \boxed{1}
\end{aligned}$$

$\square$