THE INTEGRAL TEST

Chapter 11.3

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Example 11.3.1 : Harmonic series.

Consider

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}\,.$$

Define

$$f(x) = \frac{1}{x} \quad \text{for } x \geq 0$$

so that

$$a_n = f(n), \ \text{ for } n \geq 1 \text{ integer}\,.$$

This $f(x)$ is **continuous, positive, and decreasing**.

The integral test can be applied here:

$$\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty}\left[\ln(x)\right]\Big|_1^a = \lim_{a\to\infty}\ln(a) = \infty$$

Integral diverges $\Rightarrow$ **Series diverges** .

$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty \quad \text{is } \textbf{divergent}. \tag{1}$$

Example 11.3.2

Consider

$$\sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \frac{1}{n\ \ln(n)\ \ln(\ln(n))}\,.$$

Define

$$f(x) = \frac{1}{x\ \ln(x)\ \ln(\ln(x))} \quad \text{for } x \geq 3$$

so that

$$a_n = f(n), \ \text{ for } n \geq 3 \text{ integer}\,.$$

This $f(x)$ is **continuous, positive, and decreasing**.

The integral test can be applied here:

$$\lim_{a\to\infty}\int_3^a f(x)\,dx = \lim_{a\to\infty}\left[\ln(\ln(\ln(x)))\right]\Big|_3^a$$
$$= \lim_{a\to\infty}\left[\ln(\ln(\ln(a)))\right] - \left[\ln(\ln(\ln(3)))\right]$$
$$= \infty$$

Integral diverges $\Rightarrow$ **Series diverges**.

Example 11.3.3

The **Riemann zeta-function** is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \,. \tag{2}$$

Find the domain $\mathcal{D}$ of $\zeta(x)$.

$1 \notin \mathcal{D}$ since the harmonic series diverges (see Example 11.3.1).
$0 \notin \mathcal{D}$ since the constant series of term 1 diverges.
If $x = -|x| < 0$, then $a_n = n^{-x} = n^{|x|}$ satisfies $\lim_{n\to\infty} n^{|x|} = \infty$ and the series diverges for such x. It follows that $\mathcal{D} \subset (0, \infty)$.

For $1 \neq x > 0$ let $g_x(y) = \frac{1}{y^x} = y^{-x}$.
Since g_x is continuous, positive, decreasing and

$$
\begin{aligned}
(x \neq 1) \Rightarrow \int_1^\infty g_x(y)\,dy
&= \int_1^\infty \frac{1}{y^x}\,dy \\
&= \left[\frac{y^{1-x}}{1-x}\right]\Bigg|_1^\infty \\
&= \begin{cases} \frac{1}{x-1} & : \quad x > 1 \\ \infty & : \quad x < 1 \end{cases}
\end{aligned}
$$

one concludes from the integral test that $\boxed{\mathcal{D} = (1, \infty)}$.

NOTE: $\sum_{n=1}^\infty \frac{1}{n^p}$ is called a p-**series** in the text.
Thus, the p-series converges if $p > 1$ and diverges if $p \leq 1$.

Example 11.3.4

Consider $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$, so that the general term is $a_n = \frac{1}{n^4}$.

List of the first ten **terms** a_n:

$$1,\ \frac{1}{16},\ \frac{1}{81},\ \frac{1}{256},\ \frac{1}{625},\ \frac{1}{1296},\ \frac{1}{2401},\ \frac{1}{4096},\ \frac{1}{6561},\ \frac{1}{10000}$$

List of the first ten **partial sums** s_n:

$$1,\ \frac{17}{16},\ \frac{1393}{1296},\ \frac{22369}{20736},\ \frac{14001361}{12960000},\ \frac{14011361}{12960000},$$

$$\frac{33654237761}{31116960000},\ \frac{538589354801}{497871360000},\ \frac{43631884298881}{40327580160000},\ \frac{43635917056897}{40327580160000}$$

Decimal approximation of the first ten **partial sums** s_n:

$$1.0,\ 1.0625,\ 1.07485,\ 1.07875,\ 1.08035,$$

$$1.08112,\ 1.08154,\ 1.08178,\ 1.08194,\ 1.08204$$

The known **exact value** with a decimal approximation is

$$\zeta(4) = \frac{\pi^4}{90} \approx 1.082323233711138191516003696 54$$

Example 11.3.5 Let $n \geq k \geq 1$ be integers. Then

$$k < x < k+1 \quad \Rightarrow \quad \frac{1}{k} > \frac{1}{x} > \frac{1}{k+1} \quad \Rightarrow \quad \frac{1}{k} > \int_k^{k+1} \frac{1}{x}\,dx > \frac{1}{k+1}$$

so that

$$\sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) > \sum_{k=1}^{n} \frac{1}{k+1}$$

and

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n) = \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \right) + (\ln(n+1) - \ln(n)) > 0\,.$$

The sequence with positive terms

$$\boxed{t_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n}$$

is therefore **bounded below** by 0 . Also

$$\begin{aligned}
&t_n - t_{n+1} \\
&= \{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\} \\
&- \{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1)\} \\
&= \ln(n+1) - \ln n - \frac{1}{n+1} \\
&= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx > 0\,.
\end{aligned}$$

Thus the sequence $n \mapsto t_n$ is **strictly decreasing**.

A theorem analogous to Theorem 6 in Section 11.1, dealing with **nonincreasing** sequences, shows that the following limit **exists**:

$$\boxed{\gamma \equiv \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)\ .}$$

The limit is known as the

Euler-Mascheroni constant

and has the approximate value

$$\boxed{\gamma \approx 0.5772156649015328606065120900824024310\ .}$$