COMPARISON TESTS

Chapter 11.4

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Example 11.4.1 Determine convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$$

Since

$$0 \leq \sin^2 n \leq 1$$
$$n\sqrt{n} = n^{3/2} > 0$$
$$0 \leq \frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

comparison with a **convergent** p-series for ($p = 3/2 > 1$) proves the **convergence** of this series.

Example 11.4.2 Determine convergence or divergence of:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}}.$$

Since

$$n < n+2 \qquad n+1 < n+2$$
$$n(n+1)(n+2) < (n+2)(n+2)(n+2) = (n+2)^3$$
$$\frac{1}{n(n+1)(n+2)} > \frac{1}{(n+2)^3}$$
$$\frac{1}{\sqrt[3]{n(n+1)(n+2)}} > \frac{1}{\sqrt[3]{(n+2)^3}} = \frac{1}{n+2}$$

comparison with a **divergent** p-series ($p = 1$) shows that this series is **divergent**.

Example 11.4.3

For

$$0 \leq t \leq \arccos(\frac{1}{2}) = \frac{\pi}{3}$$

we have

$$1 \geq \cos t \geq \frac{1}{2}\,.$$

Thus, for

$$0 \leq x \leq \frac{\pi}{3}$$

$$\sin x = \int_0^x \cos t\, dt \geq \int_0^x \frac{1}{2}\, dt = \frac{x}{2}$$

Therefore

$$\sin\left(\frac{1}{n}\right) \geq \frac{\left(\frac{1}{n}\right)}{2} = \frac{1}{2n}$$

Comparison with the divergent harmonic series shows that the series

$$\boxed{\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)}$$

is **divergent** .

Example 11.4.4

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges,}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \boxed{\text{diverges}}.$$

$$a_n = \frac{1}{n^2} > 0, \qquad b_n = \frac{1}{n} > 0$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

Compare with part (2) of Theorem 11: Limit Comparison Test:
Difference: By hypothesis $\sum_{n=1}^{\infty} b_n$ $\boxed{\text{converges}}$ in that test.

Example 11.4.5

Assume $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent.

Claim: $\sum_{n=1}^{\infty} \ln(1 + a_n)$ is convergent.

$$\frac{d}{dt} \ln(1+t) = \frac{1}{1+t}$$

$$\text{For } t \geq 0 \qquad 1 \geq \frac{1}{1+t}$$

Integrating, for $x > 0$:

$$x = \int_0^x 1 \, dt \geq \int_0^x \frac{1}{1+t} = \ln(1+x) > 0$$

$$x \rightsquigarrow a_n \ \Rightarrow \qquad a_n \geq \ln(1+a_n) > 0$$

Using the comparison theorem it follows that $\sum_{n=1}^{\infty} \ln(1 + a_n)$ is **convergent** .

Example 11.4.6 Assume $a_n > 0,\ b_n > 0$.
If $\sum_{n=1}^{\infty} a_n,\ \sum_{n=1}^{\infty} b_n$ are both convergent, then

$$\sum_{n=1}^{\infty} a_n b_n \quad \text{converges} \quad .$$

Proof.
Since $\sum_{n=1}^{\infty} b_n$ is convergent it follows that $\lim_{n\to\infty} b_n = 0$.
Then there exists some $N > 0$ with

$$n > N \quad \Rightarrow \quad 0 < b_n < 1 .$$

Multiplying by $a_n > 0$ one finds:

$$0 < a_n b_n < a_n \text{ for all } n > N,$$

so that $\sum_{n=1}^{\infty} a_n b_n$ converges by **Theorem 10**. □