POWER SERIES

Chapter 11
Section 7

Example 11.7.1

Find power series representation and interval of convergence for

$$f(x) = \frac{1+x^2}{1-x^2}$$

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \text{ for } |y| < 1 \quad \text{Let } y = x^2$$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

$$x^2 \frac{1}{1-x^2} = x^2 \sum_{n=0}^{\infty} x^{2n} = \sum_{n=1}^{\infty} x^{2n}$$

$$(1+x^2)\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} + \sum_{n=1}^{\infty} x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} 2\,x^{2n}$$

Radius of convergence:
$|y| < 1 \Leftrightarrow |x^2| < 1 \Leftrightarrow |x| < 1$ so that $R = 1$
Interval of convergence: $(-1, 1)$ since
$x = \pm 1 \Leftrightarrow |x| = 1 \Leftrightarrow y = 1$ divergence point

Example 11.7.2

Use of partial fraction decomposition:

$$\begin{aligned} f(x) &= \frac{3x-2}{2x^2-3x+1} \\ &= (2\,x-1)^{-1} + (x-1)^{-1} \\ &= (-1)\left((1-2\,x)^{-1} + (1-x)^{-1}\right) \\ &= -\sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} (-1)\,(1+2^n)\,x^n \end{aligned}$$

Radius of convergence:
$|2x| < 1 \Leftrightarrow |x| < \frac{1}{2} \Leftrightarrow R = \frac{1}{2}$
Interval of convergence: $(-\frac{1}{2}, \frac{1}{2})$,
since $x = \pm\frac{1}{2}$ are points of divergence

Example 11.7.3

Find the power series representation and radius of convergence for $f(x) = \frac{1}{(1+x)^3}$

$$
\begin{aligned}
f(x) &= \frac{1}{(1+x)^3} = (1+x)^{-3} \\
&= \left(\frac{1}{2}\right) \frac{d^2}{dx^2} (1+x)^{-1} \\
&= \left(\frac{1}{2}\right) \frac{d^2}{dx^2} \sum_{n=0}^{\infty} (-1)^n x^n \\
&= \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} (-1)^n \frac{d^2}{dx^2} x^n \\
&= \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} (-1)^n n(n-1) x^{n-2} \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{2} n(n-1) x^{n-2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2} (n+1)(n+2) x^n
\end{aligned}
$$

Radius of convergence: $|x| < 1 \Leftrightarrow R = 1$

Example 11.7.4

Find an indefinite integral as a power series:

$$\text{arctan}(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} y^{2n+1}, \quad |y| < 1$$

Substitute $y = x^2$, and integrate over x:

$$\begin{aligned}
\int \text{arctan}(x^2)\, dx &= \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x^2)^{2n+1} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int (x^2)^{2n+1}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{4n+2}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)} x^{4n+3} \; + C
\end{aligned}$$

Radius of convergence: $R = 1$

Exercise 10.7.5

Domain of the *Bessel function of order 1* ?

$$J_1(x) \doteq \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

Ratio test:

$$\lim_{n\to\infty} \frac{\left(\frac{|x|^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}\right)}{\left(\frac{|x|^{2n+1}}{n!(n+1)!2^{2n+1}}\right)}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{2^2(n+1)(n+2)}$$

$$= 0$$

$\therefore$ Radius of convergence: $R = \infty$

$\therefore$ Interval of convergence: $-\infty < x < \infty$

Example 11.7.6

$$J_1'(x) = \boxed{\frac{d}{dx}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!2^{2n+1}} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!2^{2n+1}} \boxed{\frac{d}{dx}} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{(2n+1)}}{n!(n+1)!2^{2n+1}} x^{2n}$$

$$(\clubsuit)\ x\, J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{n!(n+1)!2^{2n+1}} \boxed{x^{2n+1}}$$

$$J_1''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{n!(n+1)!2^{2n+1}} \boxed{\frac{d}{dx}} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{(2n)(2n+1)}}{n!(n+1)!2^{2n+1}} x^{2n-1}$$

$$(\spadesuit)\ x^2\, J_1''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{(2n)(2n+1)}}{n!(n+1)!2^{2n+1}} x^{2n+1}$$

Adding equations (♣) and (♠) :

$$x^2\, J_1''(x) + x\, J_1'(x)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{(2n+1)^2}}{n!(n+1)!2^{2n+1}}\, x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{\Big((4n^2+4n)+1\Big)}}{n!(n+1)!2^{2n+1}}\, x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{\Big(2^2 n(n+1)\Big)}}{n!(n+1)!2^{2n+1}}\, x^{2n+1} + J_1(x)$$

$$= \sum_{n=1}^{\infty} \frac{-(-1)^{n-1}}{(n-1)!n!2^{2(n-1)+1}}\, x^2 x^{2(n-1)+1} + J_1(x)$$

$$= -x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!2^{2n+1}}\, x^{2n+1} + J_1(x)$$

$$= -x^2\, J_1(x) + J_1(x)$$

$$= (1 - x^2)\, J_1(x)$$

Ordinary differential equation for $J_1(x)$:

$$\boxed{x^2\, J_1''(x) + x\, J_1'(x) + (x^2 - 1)\, J_1(x) = 0}$$

Example 11.7.7

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n$$

Interval of convergence: $\boxed{-1} \leq x \leq \boxed{1}$

Interval of absolute convergence: $-1 \leq x \leq 1$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n}{n^2} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} x^{n-1}$$

Interval of convergence: $\boxed{-1} \leq x < 1$

Interval of absolute convergence: $-1 < x < 1$

$$f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)}{n} x^{n-2} = \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) x^{n-2}$$

Interval of (absolute) convergence: $-1 < x < 1$