Chapter 12
Vectors and the
Geometry of Space

Section 3
The Dot Product

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Definition of Dot Product:
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
the **dot product** of $\mathbf{u}, \mathbf{v}$ is the number

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad (1)$$

Properties: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$; $c, d \in \mathbb{R}$. Then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= |\mathbf{u}|^2 \geq 0 \\ \mathbf{u} \cdot \mathbf{u} = 0 &\Leftrightarrow |\mathbf{u}| = 0 \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{0} \\ \mathbf{u} \cdot \mathbf{u} > 0 &\Leftrightarrow \mathbf{u} \neq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\ (c\mathbf{u}) \cdot \mathbf{v} &= c\,(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v}) \\ (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\ \mathbf{0} \cdot \mathbf{u} &= 0 \end{aligned}$$

Unit Vectors:

A vector $\mathbf{u}$ is a **unit vector** if $|\mathbf{u}| = 1$.

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle \neq \mathbf{0}$, then the **direction** of $\mathbf{u}$ is defined by

$$\text{dir}(\mathbf{u}) = \frac{\mathbf{u}}{|\mathbf{u}|}, \quad \text{implying that} \quad |\text{dir}(\mathbf{u})| = 1 \tag{2}$$

Let $\mathbf{e}$, $\mathbf{f}$ be two **unit** vectors and $t = \mathbf{e} \cdot \mathbf{f} \in \mathbb{R}$. Then

$$\begin{aligned}
0 \leq |t\,\mathbf{e} - \mathbf{f}|^2 &= (t\,\mathbf{e} - \mathbf{f}) \cdot (t\,\mathbf{e} - \mathbf{f}) \\
&= t\,\mathbf{e} \cdot (t\,\mathbf{e} - \mathbf{f}) - \mathbf{f} \cdot (t\,\mathbf{e} - \mathbf{f}) \\
&= t^2\,\mathbf{e} \cdot \mathbf{e} - t\,\mathbf{e} \cdot \mathbf{f} - t\,\mathbf{f} \cdot \mathbf{e} + \mathbf{f} \cdot \mathbf{f} \\
&= t^2 - 2t\,\mathbf{e} \cdot \mathbf{f} + 1 \\
&= (\mathbf{e} \cdot \mathbf{f})^2 - 2(\mathbf{e} \cdot \mathbf{f})(\mathbf{e} \cdot \mathbf{f}) + 1 \\
&= 1 - (\mathbf{e} \cdot \mathbf{f})^2
\end{aligned}$$

It follows that

$$\begin{array}{rcl} 0 & \leq & (\mathbf{e}\cdot\mathbf{f})^2 \leq 1 \\ -1 & \leq & \mathbf{e}\cdot\mathbf{f} \quad \leq \quad 1 \\ \mathbf{e}=\mathbf{f} & \Rightarrow & \mathbf{e}\cdot\mathbf{f}=1 \\ \mathbf{e}=-\mathbf{f} & \Rightarrow & \mathbf{e}\cdot\mathbf{f}=-1 \end{array}$$

More generally, for **nonzero** vectors $\mathbf{u} = |\mathbf{u}|\mathrm{dir}(\mathbf{u})$, $\mathbf{v} = |\mathbf{v}|\mathrm{dir}(\mathbf{v})$

$$\begin{array}{rcl} -1 & \leq & \mathrm{dir}(\mathbf{u})\cdot\mathrm{dir}(\mathbf{v}) \quad \leq \quad 1 \\ -|\mathbf{u}|\,|\mathbf{v}| & \leq & |\mathbf{u}|\mathrm{dir}(\mathbf{u})\cdot\mathrm{dir}(\mathbf{v})|\mathbf{v}| \quad \leq \quad |\mathbf{u}|\,|\mathbf{v}| \\ -|\mathbf{u}|\,|\mathbf{v}| & \leq & \mathbf{u}\cdot\mathbf{v} \quad \leq \quad |\mathbf{u}|\,|\mathbf{v}| \\ |\mathbf{u}\cdot\mathbf{v}| & \leq & |\mathbf{u}|\,|\mathbf{v}| \qquad \text{(Cauchy-Schwartz inequality)} \end{array}$$

Clearly, this inequality also holds when $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

Angle between Vectors:

Let $\mathbf{u}, \mathbf{v}$ be **nonzero** vectors.

The **angle** $\theta = \measuredangle(\mathbf{u}, \mathbf{v}) \in [0, \pi]$ between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\cos \measuredangle(\mathbf{u}, \mathbf{v}) = \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \mathrm{dir}(\mathbf{u}) \cdot \mathrm{dir}(\mathbf{v}) \tag{3}$$

Note: One can define the angle between the zero-vector and any vector to be $\frac{\pi}{2}$.

These definitions lead to the **equality**:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos \measuredangle(\mathbf{u}, \mathbf{v}) \tag{4}$$

Perpendicular (Orthogonal) Vectors:

Two vectors $\mathbf{u}, \mathbf{v}$ are **perpendicular (orthogonal)**, (in symbols $\mathbf{u} \perp \mathbf{v}$), if

$$\mathbf{u} \perp \mathbf{v} \quad \Leftrightarrow \quad \measuredangle(\mathbf{u}, \mathbf{v}) = \frac{\pi}{2} \quad \Leftrightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0 \tag{5}$$

Projections relative to a nonzero vector:

Given a **nonzero** vector **v** and any vector **u**

the **projection** of **u parallel** to **v** is

$$\text{proj}_{\mathbf{v}}^{\parallel}\,\mathbf{u} = (\mathbf{u}\cdot\text{dir}(\mathbf{v}))\,\text{dir}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v} = |\mathbf{u}|\cos\measuredangle(\mathbf{u},\mathbf{v})\,\frac{\mathbf{v}}{|\mathbf{v}|} \tag{6}$$

The **projection** of **u perpendicular** (orthogonal) to **v** is

$$\text{proj}_{\mathbf{v}}^{\perp}\,\mathbf{u} = \mathbf{u} - \text{proj}_{\mathbf{v}}^{\parallel}\,\mathbf{u} \tag{7}$$

For $\mathbf{v} \neq \mathbf{0}$ and arbitrary $\mathbf{u}$ it follows that

$$\begin{aligned}
\mathbf{u} &= \mathrm{proj}_{\mathbf{v}}^{\parallel}\mathbf{u} + \mathrm{proj}_{\mathbf{v}}^{\perp}\mathbf{u} \\
0 &= \left(\mathrm{proj}_{\mathbf{v}}^{\perp}\mathbf{u}\right)\cdot\mathbf{v} \\
\mathrm{proj}_{\mathbf{v}}^{\parallel}\mathbf{u} &\perp \mathrm{proj}_{\mathbf{v}}^{\perp}\mathbf{u} \\
\mathbf{u}\cdot\mathbf{v} &= \left(\mathrm{proj}_{\mathbf{v}}^{\parallel}\mathbf{u}\right)\cdot\mathbf{v} \\
\mathbf{u}\cdot\mathrm{dir}(\mathbf{v}) &= \left(\mathrm{proj}_{\mathbf{v}}^{\parallel}\mathbf{u}\right)\cdot\mathrm{dir}(\mathbf{v}) = |\mathbf{u}|\cos\measuredangle(\mathbf{u},\mathbf{v})
\end{aligned}$$