Chapter 12
Vectors and the Geometry of Space

Section 4
The Cross Product

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Definition of cross product:
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\ &= \langle\, u_2 v_3 - u_3 v_2,\, u_3 v_1 - u_1 v_3,\, u_1 v_2 - u_2 v_1 \,\rangle && (1) \\ &= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle && (2) \end{aligned}$$

Properties: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$; $r, s \in \mathbb{R}$. Then

$$\begin{aligned} (r\mathbf{u}) \times (s\mathbf{v}) &= (rs)\,(\mathbf{u} \times \mathbf{v}) \\ \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \\ (\mathbf{v} + \mathbf{w}) \times \mathbf{u} &= (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u}) \\ \mathbf{v} \times \mathbf{u} &= -\,(\mathbf{u} \times \mathbf{v}) \\ \mathbf{0} \times \mathbf{u} &= \mathbf{0} \end{aligned}$$

Products of standard basis vectors:

Let $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$ denote the **standard** basis vectors of $\mathbb{R}^3$.

The cross products of these vectors are listed in the table

$\times$	**i**	**j**	**k**
i	**0**	**k**	**-j**
j	**-k**	**0**	**i**
k	**j**	**-i**	**0**

Definition of triple scalar product: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= (\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle) \cdot \langle w_1, w_2, w_3 \rangle \\ &= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \cdot \langle w_1, w_2, w_3 \rangle \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\end{aligned}$$

The **triple scalar product** of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is the number

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{3}$$

Properties of the triple scalar product:

Since

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \qquad \text{it follows that}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{u}, \mathbf{v}, \mathbf{w}] &= -[\mathbf{v}, \mathbf{u}, \mathbf{w}] \\
&= [\mathbf{v}, \mathbf{w}, \mathbf{u}] \\
&= -[\mathbf{w}, \mathbf{v}, \mathbf{u}] \\
&= [\mathbf{w}, \mathbf{u}, \mathbf{v}] \\
&= -[\mathbf{u}, \mathbf{w}, \mathbf{v}]
\end{aligned}
$$

Some computations:

$$
\begin{aligned}
|\mathbf{u}|^2 \, |\mathbf{v}|^2 &= \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) \\
&= \quad u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 \\
&\quad + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 \\
&\quad + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 \\
(\mathbf{u} \cdot \mathbf{v})^2 &= (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\
&= \quad u_1 v_1 u_1 v_1 + u_1 v_1 u_2 v_2 + u_1 v_1 u_3 v_3 \\
&\quad + u_2 v_2 u_1 v_1 + u_2 v_2 u_2 v_2 + u_2 v_2 u_3 v_3 \\
&\quad + u_3 v_3 u_1 v_1 + u_3 v_3 u_2 v_2 + u_3 v_3 u_3 v_3 \\
&= u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 \\
&\quad + 2 u_1 v_1 u_2 v_2 + 2 u_1 v_1 u_3 v_3 + 2 u_2 v_2 u_3 v_3
\end{aligned}
$$

$$\begin{aligned} |\mathbf{u}\times\mathbf{v}|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2 \\ &= \quad u_2^2v_3^2-2u_2v_3u_3v_2+u_3^2v_2^2 \\ &= +u_3^2v_1^2-2u_3v_1u_1v_3+u_1^2v_3^2 \\ &= +u_1^2v_2^2-2u_1v_2u_2v_1+u_2^2v_1^2 \end{aligned}$$

The computations show that

$$|\mathbf{u}|^2\,|\mathbf{v}|^2=(\mathbf{u}\cdot\mathbf{v})^2+|\mathbf{u}\times\mathbf{v}|^2 \tag{4}$$

Since

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos\measuredangle(\mathbf{u},\mathbf{v}),\qquad 0\le\measuredangle(\mathbf{u},\mathbf{v})\le\pi$$

it follows that

$$|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2\,|\mathbf{v}|^2\left(1-\cos^2\measuredangle(\mathbf{u},\mathbf{v})\right)=|\mathbf{u}|^2\,|\mathbf{v}|^2\sin^2\measuredangle(\mathbf{u},\mathbf{v}) \tag{5}$$

Since $0 \le \measuredangle(\mathbf{u},\mathbf{v}) \le \pi$ and therefore $0 \le \sin\measuredangle(\mathbf{u},\mathbf{v}) \le 1$ one finds by taking the square root in equation (5) that

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\measuredangle(\mathbf{u},\mathbf{v}) \tag{6}$$

Let $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if
$\sin\measuredangle(\mathbf{u},\mathbf{v}) = 0$, i.e. $\measuredangle(\mathbf{u},\mathbf{v}) = 0$ or $\measuredangle(\mathbf{u},\mathbf{v}) = \pi$.
In other words, $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ precisely when $\mathbf{u}$ and $\mathbf{v}$ are either **parallel** or **antiparallel** ($\mathbf{v}$ is a nonzero multiple of $\mathbf{u}$).

Assume that $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. Then, since

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = [\mathbf{u},\mathbf{v},\mathbf{u}] = [\mathbf{u},\mathbf{u},\mathbf{v}] = -[\mathbf{u},\mathbf{u},\mathbf{v}] = 0$$

and

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = [\mathbf{u},\mathbf{v},\mathbf{v}] = [\mathbf{v},\mathbf{v},\mathbf{u}] = -[\mathbf{v},\mathbf{v},\mathbf{u}] = 0,$$

it follows that $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
Thus, $\mathbf{u}\times\mathbf{v}$ is **perpendicular** to the **plane** spanned by $\mathbf{u}$ and $\mathbf{v}$.
In terms of $\operatorname{dir}(\mathbf{u}\times\mathbf{v})$, the **direction** of $\mathbf{u}\times\mathbf{v}$, we have

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}\times\mathbf{v}|\ \operatorname{dir}(\mathbf{u}\times\mathbf{v}) = |\mathbf{u}|\,|\mathbf{v}|\sin\measuredangle(\mathbf{u},\mathbf{v})\operatorname{dir}(\mathbf{u}\times\mathbf{v}) \tag{7}$$

and (since $\mathbf{u}\times\mathbf{v}\neq\mathbf{0}$)

$$(\mathbf{u}\times\mathbf{v})\cdot\operatorname{dir}(\mathbf{u}\times\mathbf{v}) = [\mathbf{u},\mathbf{v},\operatorname{dir}(\mathbf{u}\times\mathbf{v})] = |\mathbf{u}|\,|\mathbf{v}|\sin\measuredangle(\mathbf{u},\mathbf{v}) > 0 \tag{8}$$

<u>Conclusion</u>:
The vector $\mathbf{u}\times\mathbf{v}\neq\mathbf{0}$ is characterized by the properties

Length: $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\measuredangle(\mathbf{u},\mathbf{v}) > 0$

Direction: $\operatorname{dir}(\mathbf{u}\times\mathbf{v})$ is the **unit** vector **perpendicular** to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ that satisfies $[\mathbf{u},\mathbf{v},\operatorname{dir}(\mathbf{u}\times\mathbf{v})] > 0$.