Chapter 13
Vectors-Valued Functions
and
Motions in Space

Section 5
Torsion and the Unit Binormal Vector **B**

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Basic definitions:

Let $\mathbf{r}(t)$ describe some smooth motion in space.

The basic kinematic quantities are defined as follows:

Velocity $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t)$

Speed $v(t) = |\mathbf{v}(t)| = \sqrt{(\mathbf{v} \cdot \mathbf{v})}(t)$

Acceleration $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}(t) = \frac{d^2\mathbf{r}}{dt^2}(t)$

Arc length $\frac{ds}{dt}(t) = v(t)$ or $ds = v(t)\,dt$

T, N and curvature κ:

$$\begin{aligned}
\frac{d\mathbf{r}}{ds} &= \frac{1}{v}\,\frac{d\mathbf{r}}{dt} = \frac{1}{v}\,\mathbf{v} = \mathbf{T}, \quad |\mathbf{T}| = 1 \\
\frac{d^2\mathbf{r}}{ds^2} &= \frac{d\mathbf{T}}{ds} = \frac{1}{v}\,\frac{d\mathbf{T}}{dt} = \frac{1}{v}\,\frac{d}{dt}\left(\frac{1}{v}\,\frac{d\mathbf{r}}{dt}\right) = \kappa\,\mathbf{N}, \quad \kappa \geq 0, |\mathbf{N}| = 1 \\
\kappa &= \left|\frac{d^2\mathbf{r}}{ds^2}\right| = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{1}{v}\,\frac{d\mathbf{T}}{dt}\right| = \left|\frac{1}{v}\,\frac{d}{dt}\left(\frac{1}{v}\,\frac{d\mathbf{r}}{dt}\right)\right| \\
\frac{d}{dt}\left(v^2\right) &= \frac{d}{dt}\,(\mathbf{v}\cdot\mathbf{v}) \\
2v\,\frac{dv}{dt} &= 2\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = 2\mathbf{a}\cdot\mathbf{v} \\
\frac{dv}{dt} &= \frac{1}{v}\,\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{1}{v}\,\mathbf{v}\cdot\mathbf{a} = \mathbf{T}\cdot\mathbf{a} \\
\frac{d}{dt}\left(\frac{1}{v}\right) &= -\frac{1}{v^2}\,\frac{dv}{dt} = -\frac{1}{v^3}\,\mathbf{v}\cdot\mathbf{a} = -\frac{1}{v^2}\,\mathbf{T}\cdot\mathbf{a}
\end{aligned}$$

Acceleration $\mathbf{a}$ and curvature κ:

Note that $\mathbf{T} \cdot \mathbf{T} = 1$ implies $\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = (\kappa \mathbf{N} \cdot \mathbf{T}) = 0$ so that $\mathbf{N} \cdot \mathbf{T} = 0$ if $\kappa > 0$.

$$
\begin{aligned}
\kappa \mathbf{N} &= \frac{1}{v} \frac{d}{dt} \left(\frac{1}{v} \frac{d\mathbf{r}}{dt} \right) = \frac{1}{v} \frac{d}{dt} \left(\frac{\mathbf{v}}{v} \right) \\
&= \frac{1}{v} \left(\frac{d}{dt} \left(\frac{1}{v} \right) \mathbf{v} + \frac{1}{v} \frac{d\mathbf{v}}{dt} \right) \\
&= \frac{1}{v} \left(-\frac{1}{v^2} (\mathbf{T} \cdot \mathbf{a}) \, \mathbf{v} + \frac{1}{v} \mathbf{a} \right) \\
&= \frac{1}{v^2} (\mathbf{a} - (\mathbf{a} \cdot \mathbf{T}) \, \mathbf{T}) \\
\kappa &= \frac{1}{v^2} |\mathbf{a} - (\mathbf{a} \cdot \mathbf{T}) \, \mathbf{T}|
\end{aligned}
$$

Tangential and normal acceleration:

$$\begin{aligned}
\mathbf{a} &= \mathbf{a}_{tangential} + \mathbf{a}_{normal} = a_T\,\mathbf{T} + a_N\,\mathbf{N} \\
&= (\mathbf{a}\cdot\mathbf{T})\ \mathbf{T} + \left(v^2\kappa\right)\ \mathbf{N} \\
\mathbf{a}_{tangential} &= a_T\,\mathbf{T} = (\mathbf{a}\cdot\mathbf{T})\ \mathbf{T} \\
\mathbf{a}_{normal} &= a_N\,\mathbf{N} = (\mathbf{a}\cdot\mathbf{N})\ \mathbf{N} \\
a_T &= \mathbf{a}\cdot\mathbf{T} \\
a_N &= \mathbf{a}\cdot\mathbf{N} = v^2\kappa \\
|\mathbf{a}|^2 &= a_T^2 + a_N^2 = (\mathbf{a}\cdot\mathbf{T})^2 + v^4\,\kappa^2 \\
\kappa &= \frac{\sqrt{|\mathbf{a}|^2 - (\mathbf{a}\cdot\mathbf{T})^2}}{|\mathbf{v}|^2} = \frac{\sqrt{|\mathbf{a}|^2 - |\mathbf{a}|^2\,\cos^2\measuredangle(\mathbf{a},\mathbf{v})}}{|\mathbf{v}|^2} \\
&= \frac{|\mathbf{a}|\sin\measuredangle(\mathbf{a},\mathbf{v})}{|\mathbf{v}|^2} = \frac{|\mathbf{a}||\mathbf{v}|\sin\measuredangle(\mathbf{a},\mathbf{v})}{|\mathbf{v}|^3} = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}
\end{aligned}$$

The binormal vector $\mathbf{B}$:

The **binormal** vector is defined as the **unit** vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{d\mathbf{r}}{ds} \times \frac{1}{\kappa} \frac{d^2\mathbf{r}}{ds^2} = \frac{1}{\kappa} \frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} .$$

The ordered triple of unit vectors $< \mathbf{T}, \mathbf{N}, \mathbf{B} >$ is analogous to the triple $< \mathbf{i}, \mathbf{j}, \mathbf{k} >$. For a given time t (or arc length s) these vectors could be used as the basis vectors of a **righthanded Cartesian** coordinate system. In general, this system will change with time t (or arc length s). It is known as the co-moving **TNB**-frame or **Frenet** frame of the curve.

Rates of change with respect to arc length s:

$$\frac{d\mathbf{T}}{ds} = \kappa\,\mathbf{N}$$

$$\mathbf{N}\cdot\mathbf{N} = 1 \;\Rightarrow\; \frac{d\mathbf{N}}{ds}\cdot\mathbf{N} = 0$$

$$\frac{d\mathbf{N}}{ds} = \alpha\,\mathbf{T} + \tau\,\mathbf{B}$$

$$\mathbf{N}\cdot\mathbf{T} = 0 \;\Rightarrow\; \frac{d\mathbf{N}}{ds}\cdot\mathbf{T} + \mathbf{N}\cdot\frac{d\mathbf{T}}{ds} = 0$$

$$\alpha = \frac{d\mathbf{N}}{ds}\cdot\mathbf{T} = -\mathbf{N}\cdot\frac{d\mathbf{T}}{ds} = -\mathbf{N}\cdot(\kappa\,\mathbf{N}) = -\kappa$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$$

$$
\begin{aligned}
\frac{d(\mathbf{T} \times \mathbf{N})}{ds} &= \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\
&= (\kappa\,\mathbf{N}) \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\
&= \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\
&= \mathbf{T} \times (-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) \\
&= \tau\,\mathbf{T} \times \mathbf{B} \\
\frac{d\mathbf{B}}{ds} &= -\tau\,\mathbf{N}
\end{aligned}
$$

Frenet formulae :

$$\frac{d\mathbf{T}}{ds} = \kappa\,\mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau\,\mathbf{N}$$

Formula for the **torsion** τ :

$$
\begin{aligned}
\tau &= -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} \\
&= -\frac{d}{ds}\left(\frac{1}{\kappa}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right)\right) \cdot \left(\frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2}\right) \\
&= -\left(\frac{1}{\kappa}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right)\right) \cdot \left(\frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2}\right) \\
&= -\frac{1}{\kappa^2}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) \cdot \frac{d^2\mathbf{r}}{ds^2} \\
&= \frac{1}{\kappa^2}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3} \\
\tau &= \frac{1}{\kappa^2}\left[\frac{d\mathbf{r}}{ds}, \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3}\right]
\end{aligned}
$$

Alternate formula for the **torsion** τ :

$$\begin{aligned}
\tau &= \frac{1}{\kappa^2}\left[\frac{d\mathbf{r}}{ds}, \frac{d^2\mathbf{r}}{ds^2}, \frac{d^3\mathbf{r}}{ds^3}\right] \\
\kappa^2 &= \frac{|\mathbf{v}\times\mathbf{a}|^2}{v^6} \\
\frac{d\mathbf{r}}{ds} &= \frac{1}{v}\frac{d\mathbf{r}}{dt} \\
\frac{d^2\mathbf{r}}{ds^2} &= \frac{1}{v}\frac{d}{dt}\left(\frac{1}{v}\frac{d\mathbf{r}}{dt}\right) = \frac{1}{v^2}\frac{d^2\mathbf{r}}{dt^2} + \ldots \\
\frac{d^3\mathbf{r}}{ds^3} &= \frac{1}{v}\frac{d}{dt}\left(\frac{1}{v}\frac{d}{dt}\left(\frac{1}{v}\frac{d\mathbf{r}}{dt}\right)\right) = \frac{1}{v^3}\frac{d^3\mathbf{r}}{dt^3} + \ldots
\end{aligned}$$

The terms not given explicitly above (indicated by ...) do not contribute to the formula for τ. The **torsion** τ expressed in terms of a Triple Scalar Product (3 by 3 determinant) is given by

$$\tau \;=\; \frac{v^6}{|\mathbf{v}\times\mathbf{a}|^2}\left[\frac{1}{v}\frac{d\mathbf{r}}{dt}\,,\;\frac{1}{v^2}\frac{d^2\mathbf{r}}{dt^2}\,,\;\frac{1}{v^3}\frac{d^3\mathbf{r}}{dt^3}\right]$$

or finally

$$\tau \;=\; \frac{\left[\frac{d\mathbf{r}}{dt}\,,\;\frac{d^2\mathbf{r}}{dt^2}\,,\;\frac{d^3\mathbf{r}}{dt^3}\right]}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\left[\frac{d\mathbf{r}}{dt}\,,\;\frac{d^2\mathbf{r}}{dt^2}\,,\;\frac{d^3\mathbf{r}}{dt^3}\right]}{\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right|^2} = \frac{\left[\mathbf{v}\,,\;\mathbf{a}\,,\;\frac{d\mathbf{a}}{dt}\right]}{|\mathbf{v}\times\mathbf{a}|^2}$$