Section 7.4

Integration of Rational Functions by Partial Fractions

Part 1

Algebra Review 1

A **real polynomial** of degree n is an expression of the form:

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} a_k \cdot x^k \\ &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \end{aligned}$$

where the a_k are real numbers and $a_n \neq 0$

A **real rational function** is an expression of the form: $f(x) = \frac{P(x)}{Q(x)}$, where $P(x), Q(x) \neq 0$ are real polynomials.

$f(x)$ is **proper** if $deg(P(x)) < deg(Q(x))$.

$f(x)$ is **improper** if $deg(P(x)) \geq deg(Q(x))$.

Algebra Review 2

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$,
there are **unique** polynomials $R(x)$, $S(x)$
such that

$$f(x) = S(x) + \frac{R(x)}{Q(x)} \text{ and } deg(R(x)) < deg(Q(x)).$$

Note: $deg(R(x)) = -\infty$ if $R(x) \equiv 0$.

If the given rational function is improper, do a long division to decompose it as the sum of the polynomial $S(x)$ and the proper rational function $\frac{R(x)}{Q(x)}$ first.

The integral of $S(x)$ is easy to do.

Focus on the part $\frac{R(x)}{Q(x)}$ from now on.

Algebra Review 3

Factor the denominator $Q(x)$ completely into

- linear real polynomials
- quadratic irreducible real polynomials

Write the function as the sum of its **partial fractions**.

One distinguishes the following **four** cases:

Case I

$Q(x)$ is a product of **distinct linear factors:**

$$Q(x) = \prod_{j=1}^{k} \left(a_j x + b_j\right)$$
$$= (a_1 x + b_1) \cdot (a_2 x + b_2) \cdot \ldots \cdot (a_k x + b_k) \, .$$

Then

$$\frac{R(x)}{Q(x)} = \sum_{j=1}^{k} \frac{A_j}{a_j x + b_j}$$
$$= \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \ldots + \frac{A_k}{a_k x + b_k}.$$

Case II

$Q(x)$ is a product of linear factors
some of which are repeated:

If $Q(x)$ contains a factor, where $r > 1$

$$\prod_{l=1}^{r} (ax + b)$$
$$= (ax + b)^r \,,$$

then $\frac{R(x)}{Q(x)}$ contains a summand of the form

$$\sum_{l=1}^{r} \frac{A_l}{(ax+b)^l}$$
$$= \frac{A_1}{(ax+b)^1} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}.$$

Case III

$Q(x)$ contains irreducible quadratic factors,
none of which is repeated:

If $Q(x)$ contains a factor of the form

$$ax^2 + bx + c \text{ with } b^2 - 4ac < 0,$$

then $\frac{R(x)}{Q(x)}$ contains a summand of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Case IV

$Q(x)$ contains
a repeated irreducible quadratic factor:

If $Q(x)$ contains a factor of the form

$$\left(ax^2+bx+c\right)^r \text{ with } b^2-4ac<0,\ r>1$$

then $\frac{R(x)}{Q(x)}$ contains a summand of the form

$$\frac{A_1x+B_1}{\left(ax^2+bx+c\right)^1} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \dots + \frac{A_rx+B_r}{\left(ax^2+bx+c\right)^r}$$