Section 7.4

Integration of Rational Functions by Partial Fractions

Part 2

Case I

The terms in the partial fraction decomposition of the integrand are of the form $\frac{1}{ax+b}$ $(a \neq 0)$.

Antiderivative:

$$\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + C$$

Case II

The terms in the partial fraction decomposition of the integrand are of the form $\frac{1}{(ax+b)^k}$ $(a \neq 0,\ k \geq 2)$.

Antiderivative:

$$\int \frac{1}{(ax+b)^k}\ dx = \frac{(ax+b)^{-k+1}}{a\,(-k+1)} + C$$

Case III

The terms in the partial fraction decomposition of the integrand are of the form $\frac{1}{ax^2+bx+c}$ ($a \neq 0$).

Antiderivative:

$$\int \frac{1}{ax^2+bx+c}\, dx =$$

$$2 \arctan\left(\frac{2\,ax+b}{\sqrt{4\,ca-b^2}}\right)\frac{1}{\sqrt{4\,ca-b^2}} + C$$

Case IV

The terms in the partial fraction decomposition of the integrand are of the form $\frac{1}{(ax^2+bx+c)^k}$ $\left(a \neq 0,\ k \geq 2, \text{ for example } k = 2\right)$.

$$\int \frac{1}{(ax^2+bx+c)^2}\, dx =$$

$$\frac{2\,ax+b}{\left(4\,ca-b^2\right)\left(ax^2+bx+c\right)}+$$

$$4\,a\arctan\left(\frac{2\,ax+b}{\sqrt{4\,ca-b^2}}\right)\left(4\,ca-b^2\right)^{-3/2}+C$$

Case IV - continued: $k = 3$

$$\int \frac{1}{(ax^2 + bx + c)^3}\, dx =$$

$$1/2\, \frac{2\, ax + b}{\left(4\, ca - b^2\right)\left(ax^2 + bx + c\right)^2} + 6\, \frac{a^2 x}{\left(4\, ca - b^2\right)^2\left(ax^2 + bx + c\right)} +$$

$$3\, \frac{ab}{\left(4\, ca - b^2\right)^2\left(ax^2 + bx + c\right)} + 12\, a^2 \arctan\left(\frac{2\, ax + b}{\sqrt{4\, ca - b^2}}\right)\left(4\, ca - b^2\right)^{-5/2} + C$$

Problem 17

$$\int \frac{x^2}{x+1}\,dx =$$

$$\int (x - 1 + (x+1)^{-1})\,dx =$$

$$(1/2)\ x^2 - x + \ln|x+1| + C$$

Problem 24

$$\int \frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12}\, dx =$$

$$\int (x - 1/7\ (x+4)^{-1} + 1/7\ (x-3)^{-1})\, dx =$$

$$(1/2)\ x^2 - 1/7\ \ln|x+4| + 1/7\ \ln|x-3| + C$$

$$\int_0^2 \frac{x^3 + x^2 - 12x + 1}{x^2 + x - 12}\, dx = 2 + \frac{1}{7} \ln \frac{2}{9}$$

Problem 66

$$x^4 + 1 =$$

$$(x^4 + 2x^2 + 1) - 2x^2 =$$

$$(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{A'x + B'}{x^2 - \sqrt{2}x + 1}$$

$$(Ax + B)\left(x^2 - \sqrt{2}x + 1\right) + \left(A'x + B'\right)\left(x^2 + \sqrt{2}x + 1\right) =$$

$$\left(A + A'\right)x^3 + \left(-A\sqrt{2} + B + B' + A'\sqrt{2}\right)x^2 +$$

$$\left(-B\sqrt{2} + A + A' + B'\sqrt{2}\right)x + \left(B + B'\right)$$

Problem 66 continued

This system has the solution

$$A = -A^{'} = \frac{\sqrt{2}}{4}, B = B^{'} = \frac{1}{2}$$

Therefore

$$\int \frac{1}{x^4+1}\ dx =$$

$$\frac{\sqrt{2}}{8}\ln(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1})+$$

$$\frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1)+\frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1)+C$$

Simplest forms for case IV

- $$\int \left(x^2+1\right)^{-1} dx = \text{arctan}(x)$$

- $$\int \left(x^2+1\right)^{-2} dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\,\text{arctan}(x)$$

- $$\int \left(x^2+1\right)^{-3} dx = \frac{1}{4}\frac{x}{\left(x^2+1\right)^2} + \frac{3}{8}\frac{x}{x^2+1} + \frac{3}{8}\,\text{arctan}(x)$$