TRIGONOMETRIC SUBSTITUTION

CHAPTER 7.3

Exercise 7.3.2

$$\boxed{\int_0^2 x^3\sqrt{4-x^2}\,dx}$$

Substitute $x \doteq 2\sin\theta,\ \ dx = 2\cos\theta\,d\theta$

$$= \int_0^{\pi/2} 2^3\sin^3\theta \cdot 2\cos\theta \cdot 2\cos\theta\,d\theta$$

$$= 2^5\int_0^{\pi/2}\left(1-\cos^2\theta\right)\cdot\cos^2\theta\ \ \sin\theta\,d\theta$$

Substitute $u \doteq \cos\theta,\ \ du = (-1)\sin\theta\,d\theta$

$$= 32\int_1^0\left(1-u^2\right)u^2(-1)\,du$$

$$= \left[32\frac{1}{3}u^3 - 32\frac{1}{5}u^5\right]\Bigg|_0^1$$

$$= 32\frac{1}{3} - 32\frac{1}{5}$$

$$= \boxed{\frac{64}{15}}$$

Exercise 7.3.9

$$\boxed{\int \frac{dx}{x^3\sqrt{x^2-16}}}$$

Substitute $x \doteq 4\sec\theta,\ dx = 4\sec\theta\ \tan\theta\, d\theta$

$$= \int \frac{4\sec\theta\ \tan\theta\, d\theta}{(4\sec\theta)^3\sqrt{(4\sec\theta)^2-16}}$$

$$= 4^{-3}\int \frac{\tan\theta\, d\theta}{\sec^2\theta\sqrt{\tan^2\theta}}$$

$$= 4^{-3}\int \cos^2\theta\, d\theta$$

$$= 4^{-3}\int \frac{1}{2}(1+\cos(2\theta))$$

$$= 4^{-3}\frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$= \frac{1}{128}\left(\arccos\left(\frac{4}{x}\right) + \frac{1}{2}2\sin\theta\ \cos\theta\right) + C$$

$$= \boxed{\frac{1}{128}\arccos(\frac{4}{x}) + \frac{1}{128}\frac{\sqrt{x^2-16}}{x}\frac{4}{x} + C}$$

Exercise 7.3.21

$$\boxed{\int \sqrt{2x - x^2}\, dx}$$

$$= \int \sqrt{1 - \left(1 - 2x + x^2\right)}\, dx$$

$$= \int \sqrt{1 - (x-1)^2}\, dx$$

Substitute $x - 1 \doteq \sin\theta,\ \ dx = \cos\theta\, d\theta$

$$= \int \cos\theta\ \cos\theta\, d\theta$$

$$= \int \frac{1}{2}(1 + \cos(2\theta))\ d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

$$= \frac{1}{2}(\theta + \sin\theta\ \cos\theta) + C$$

$$= \frac{1}{2}\left(\arcsin(x-1) + (x-1)\sqrt{1-(x-1)^2}\right) + C$$

$$= \boxed{\frac{1}{2}\left(\arcsin(x-1) + (x-1)\sqrt{2x-x^2}\right) + C}$$

Exercise 7.3.30 Version (a)

$$\boxed{\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx}$$

(a) $x \doteq a\tan\theta,\ \ dx = a\sec^2\theta\,d\theta$

$$
\begin{aligned}
&= \int \frac{\tan^2\theta}{\sec^3\theta}\,\sec^2\theta\,d\theta \\
&= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta \\
&= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\
&= \int (\sec\theta - \cos\theta)\,d\theta \\
&= \ln|\sec\theta + \tan\theta| - \sin\theta + C \\
&= \boxed{\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{a^2+x^2}} + C}
\end{aligned}
$$

Exercise 7.3.30 Version (b)

$$\boxed{\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx}$$

(b) $x \doteq a \sinh t,\ \ dx = a \cosh t\, dt$

$$\begin{aligned}
&= \int \frac{\sinh^2 t}{\cosh^3 t}\ \cosh t\, dt \\
&= \int \frac{\sinh^2 t}{\cosh^2 t}\, dt \\
&= \int \frac{\cosh^2 t\ -1}{\cosh^2 t}\, dt \\
&= \int \left(1 - \frac{1}{\cosh^2 t}\right)\, dt \\
&= t - \tanh t + C \\
&= \boxed{\sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{x^2+a^2}} + C}
\end{aligned}$$

Exercise 7.3.34

$$\boxed{\int \frac{dx}{x^4\sqrt{x^2-2}}}$$

Substitute $x \doteq \sqrt{2}\sec\theta,\ dx = \sqrt{2}\sec\theta\ \tan\theta\, d\theta$

$$= \frac{1}{2^2}\int \frac{\sec\theta\ \tan\theta\, d\theta}{\sec^4\theta\ \tan\theta}$$
$$= \frac{1}{4}\int \frac{d\theta}{\sec^3\theta}$$
$$= \frac{1}{4}\int \cos^3\theta\, d\theta$$
$$= \frac{1}{4}\int \left(1-\sin^2\theta\right)\cos\theta\, d\theta$$

Substitute $u \doteq \sin\theta,\;\; du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \left(1 - u^2\right)\, du$$

$$= \frac{1}{4}\left(u - \frac{1}{3}u^3\right) + C$$

$$= \boxed{\frac{\sqrt{x^2-2}}{4x} - \frac{\sqrt{\left(x^2-2\right)^3}}{12x^3} + C}$$

Exercise 7.3.37 $0 < r < R$

A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis.

Cylindrical shells with symmetry $x \leftrightarrow -x$:

$$\boxed{Volume \ = 2\int_{R-r}^{R+r} (2\pi y)\sqrt{r^2 - (y-R)^2}\,dy}$$

Substitute $u \doteq y - R$

$$= 4\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\,du$$

(Symmetry $u \leftrightarrow -u \ \Rightarrow$ first integral $= 0$)

$$= (4\pi R) \int_{-r}^{r} \sqrt{r^2 - u^2}\,du$$

(Integral equals area of halfdisk of radius r)

$$= (4\pi R)(\frac{1}{2}\pi r^2)$$

$$= \boxed{2\pi^2 R r^2}$$

Exercise 7.3.38 Section 11.6, Figure 1

Example 2 (p. 430):

Ellipse: $x^2/a^2 + y^2/b^2 = 1$ Area $= \pi ab$

Ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ $Volume$?

For fixed z with $-c \le z \le c$ the ellipsoid gives an ellipse described by $x^2/a^2 + y^2/b^2 = 1 - z^2/c^2$, whose area is
$\pi \left(a\sqrt{1 - z^2/c^2}\right)\left(b\sqrt{1 - z^2/c^2}\right) = \pi ab\left(1 - z^2/c^2\right)$

$$\boxed{Volume = \int_{-c}^{c} \pi ab \left(1 - \frac{z^2}{c^2}\right) dz}$$

$$= \pi ab \left(2c - \frac{1}{3}\frac{z^3}{c^2}\Bigg|_{-c}^{c}\right)$$

$$= \pi ab \left(2c - \frac{2}{3}c\right)$$

$$= \boxed{\frac{4}{3}\pi abc}$$