COMPLEX NUMBERS

Appendix 5
Definition:
A complex number is a pair \((a, b) \in \mathbb{R} \times \mathbb{R}\).

\(a\) is called the **real part** of the complex number \((a, b)\).
\(b\) is called the **imaginary part** of the complex number \((a, b)\).

The **set** of complex numbers is denoted by \(\mathbb{C}\).

Often a single letter is used to denote a complex number, such as \(c = (a, b) \in \mathbb{C}\).
Some special complex numbers:

**complex zero** \( 0_C = (0_R, 0_R) \)

**complex one** \( 1_C = (1_R, 0_R) \)

**imaginary unit** \( i = \sqrt{-1_R} = (0_R, 1_R) \)
Operations:

Equality \( (a, b) = (c, d) \) if \( a = c \) and \( b = d \)

Addition \( (a, b) + (c, d) = (a + c, b + d) \)

Negative \( -(a, b) = (-a, -b) \)

Subtraction \( (a, b) - (c, d) = (a, b) + (-c, d)) = (a - c, b - d) \)

Multiplication \( (a, b) \ast (c, d) = (ac - bd, ad + bc) \)

Addition and multiplication are commutative, associative and distributive.
More operations:

**Reciprocal**  
If \((a, b) \neq 0_\mathbb{C}\) then \((a, b)^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)\)

**Quotient**  
If \((c, d) \neq 0_\mathbb{C}\) then \(\frac{(a, b)}{(c, d)} = (a, b) \ast (c, d)^{-1}\)

**Absolute value**  
\(|(a, b)| = \sqrt{a^2 + b^2} \geq 0\)

**Complex conjugate**  
\(\overline{(a, b)} = (a, -b)\)
Polar coordinates:

**Polar radius** \( r = |(a, b)| \)

**Polar angle** \( \theta = \arctan \left( \frac{b}{a} \right) \equiv \arg (a, b) \)

\( \mathbb{R} \) polar form \( (a, b) = r e^{i\theta} = r (\cos \theta + i \sin \theta) \)

\( \mathbb{C} \) polar form \( (a, b) = |(a, b)| \exp (i \arg (a, b)) \)

\( \mathbb{C} \) logarithm \( \ln(a, b) = \ln(|(a, b)|) + i \arg (a, b) \)
More about \( \mathbb{C} \) logarithms: Let \((a, b), (c, d)\) be non-zero. The logarithm is an inverse function to the exponential in the sense that

\[
(a, b) \neq (0, 0) \implies (a, b) = \exp(\ln (a, b)) = e^{\ln (a,b)}.
\] (1)

Under logarithms \textbf{products} are converted to \textbf{sums}, since

\[
(a, b) \cdot (c, d) = |(a, b) \cdot (c, d)| \cdot \exp(i \arg((a, b) \cdot (c, d)))
\]

\[
= |(a, b)| \cdot \exp(i \arg (a, b)) \cdot |(c, d)| \cdot \exp(i \arg (c, d))
\]

\[
= |(a, b)| \cdot |(c, d)| \cdot \exp(i \arg (a, b)) \cdot \exp(i \arg (c, d))
\]

\[
= |(a, b)| \cdot |(c, d)| \cdot \exp(i (\arg (a, b) + \arg (c, d)))
\]

implies that

\[
|(a, b) \cdot (c, d)| = |(a, b)| \cdot |(c, d)|
\]

\[
\arg((a, b) \cdot (c, d)) = \arg (a, b) + \arg (c, d) + n2\pi, \quad n \in \mathbb{Z}.
\]
Hence

\[ |(a, b) \cdot (c, d)| = |(a, b)| \cdot |(c, d)| \]
\[ \arg ((a, b) \cdot (c, d)) = \arg (a, b) + \arg (c, d) + n2\pi, \quad n \in \mathbb{Z} \]
\[ \ln ((a, b) \cdot (c, d)) = \ln |(a, b)| \cdot |(c, d)| + i \arg ((a, b) \cdot (c, d)) \]
\[ = \ln (|(a, b)| \cdot |(c, d)|) + i(\arg (a, b) + \arg (c, d)) \]
\[ = \ln |(a, b)| + \ln |(c, d)| + i \arg (a, b) + i \arg (c, d) \]
\[ = (\ln |(a, b)| + i(\arg (a, b))) + (\ln |(c, d)| + i \arg (c, d)) \]
\[ = \ln (a, b) + \ln (c, d) \]

Consequently, (up to \( n \cdot 2\pi i, \ n \in \mathbb{Z} \))

\[ \ln ((a, b) \cdot (c, d)) = \ln |(a, b)| \cdot |(c, d)| + i \arg ((a, b) \cdot (c, d)) \]
\[ = \ln (|(a, b)| \cdot |(c, d)|) + i(\arg (a, b) + \arg (c, d)) \]
\[ = \ln |(a, b)| + \ln |(c, d)| + i \arg (a, b) + i \arg (c, d) \]
\[ = (\ln |(a, b)| + i(\arg (a, b))) + (\ln |(c, d)| + i \arg (c, d)) \]
\[ = \ln (a, b) + \ln (c, d) \]

The **logarithm of a product** is the **sum of the logarithms of the factors** (up to \( n \cdot 2\pi i, \ n \in \mathbb{Z} \)).