COMPLEX NUMBERS

Appendix 5

Definition:
A **complex** number is a pair $(a, b) \in \mathbb{R} \times \mathbb{R}$.

a is called the **real part** of the complex number (a, b).
b is called the **imaginary part** of the complex number (a, b).

The **set** of complex numbers is denoted by $\mathbb{C}$.

Often a single letter is used to denote a complex number, such as $c = (a, b) \in \mathbb{C}$.

Some special complex numbers:

complex zero $0_{\mathbb{C}} = (0_{\mathbb{R}}, 0_{\mathbb{R}})$

complex one $1_{\mathbb{C}} = (1_{\mathbb{R}}, 0_{\mathbb{R}})$

imaginary unit $i = \sqrt{-1_{\mathbb{R}}} = (0_{\mathbb{R}}, 1_{\mathbb{R}})$

Operations:

Equality $(a, b) = (c, d)$ if $a = c$ and $b = d$

Addition $(a, b) + (c, d) = (a + c, b + d)$

Negative $-(a, b) = (-a, -b)$

Subtraction $(a, b) - (c, d) = (a, b) + (-(c, d)) = (a - c, b - d)$

Multiplication $(a, b) * (c, d) = (ac - bd, ad + bc)$

Addition and multiplication are **commutative**, **associative** and **distributive**.

More operations:

Reciprocal If $(a,b) \neq 0_{\mathbb{C}}$ then $(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Quotient If $(c,d) \neq 0_{\mathbb{C}}$ then $\frac{(a,b)}{(c,d)} = (a,b) * (c,d)^{-1}$

Absolute value $|(a,b)| = \sqrt{a^2+b^2} \geq 0$

Complex conjugate $\overline{(a,b)} = (a,-b)$

Polar coordinates:

Polar radius $r = |(a,b)|$

Polar angle $\theta = \text{arctan}\left(\frac{b}{a}\right) \equiv \text{arg}\,(a,b)$

$\mathbb{R}$ **polar form** $(a,b) = r\,e^{i\theta} = r\,(\cos\theta + i\,\sin\theta)$

$\mathbb{C}$ **polar form** $(a,b) = |(a,b)|\,\exp\,(i\,\text{arg}\,(a,b))$

$\mathbb{C}$ **logarithm** $\ln(a,b) = \ln(|(a,b)|) + i\,\text{arg}\,(a,b)$

More about $\mathbb{C}$ logarithms: Let $(a,b)\,,(c,d)$ be **non-zero**. The logarithm is an inverse function to the exponential in the sense that

$$(a,b) \neq (0,0) \quad \Rightarrow \quad (a,b) = \exp(\ln(a,b)) = e^{\ln(a,b)}\,. \tag{1}$$

Under logarithms **products** are converted to **sums**, since

$$\begin{aligned}
(a,b)\cdot(c,d) &= |(a,b)\cdot(c,d)|\cdot\exp(i\arg((a,b)\cdot(c,d))) \\
&= |(a,b)|\cdot\exp(i\arg(a,b))\cdot|(c,d)|\cdot\exp(i\arg(c,d)) \\
&= |(a,b)|\cdot|(c,d)|\cdot\exp(i\arg(a,b))\cdot\exp(i\arg(c,d)) \\
&= |(a,b)|\cdot|(c,d)|\cdot\exp(i(\arg(a,b)+\arg(c,d)))
\end{aligned}$$

implies that

$$\begin{aligned}
|(a,b)\cdot(c,d)| &= |(a,b)|\cdot|(c,d)| \\
\arg((a,b)\cdot(c,d)) &= \arg(a,b)+\arg(c,d)+n2\pi, \quad n\in\mathbb{Z}\,.
\end{aligned}$$

Hence

$$
\begin{aligned}
|(a,b)\cdot(c,d)| &= |(a,b)|\cdot|(c,d)| \\
\text{arg}\,((a,b)\cdot(c,d)) &= \text{arg}\,(a,b)+\text{arg}\,(c,d)+n2\pi, \qquad n\in\mathbb{Z} \\
\ln\,((a,b)\cdot(c,d)) &= \ln\,(a,b)+\ln\,(c,d)\,.
\end{aligned}
$$

Consequently, (up to $n\cdot 2\pi i,\ n\in\mathbb{Z}$)

$$
\begin{aligned}
\ln((a,b)\cdot(c,d)) &= \ln|(a,b)\cdot(c,d)|+i\,\text{arg}\,((a,b)\cdot(c,d)) \\
&= \ln(|(a,b)|\cdot|(c,d)|)+i(\text{arg}\,(a,b)+\text{arg}\,(c,d)) \\
&= \ln\,|(a,b)|+\ln\,|(c,d)|+i\,\text{arg}\,(a,b)+i\,\text{arg}\,(c,d) \\
&= (\ln\,|(a,b)|+i(\text{arg}\,(a,b))+(\ln\,|(c,d)|+i\,\text{arg}\,(c,d)) \\
&= \ln\,(a,b)+\ln\,(c,d)
\end{aligned}
$$

The **logarithm of a product** is the **sum of the logarithms of the factors** (up to $n\cdot 2\pi i,\ n\in\mathbb{Z}$).