First Midterm Exam – Solutions

1. (15 pts) Decide for each of the following statements if they are true or not. If yes, give a short proof why. If not, give a counterexample.

   (a) The sum of two rational numbers is rational.
   TRUE. Let \( q, r \) be rational. Then there exist integers \( a, b, c, d \) with \( b, d \neq 0 \) such that \( q = \frac{a}{b} \) and \( r = \frac{c}{d} \). Then \( q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \). Since \( ad + cb \) and \( bd \) are integers we know \( q + r \) is a rational number.

   (b) The product of two irrational numbers is irrational.
   FALSE. E.g. take \( \sqrt{2} \cdot (-\sqrt{2}) = -2 \).

   (c) A bounded set of real numbers always contains a maximal element.
   FALSE. E.g. \( S = (0, 1) \) is bounded but \( \sup S = 1 \notin S \).

2. (20 pts)

   (a) Give the precise definition of the supremum of a set of real numbers.

   A number \( B \) is called a supremum of a nonempty set \( S \) of real numbers if \( B \) has the following two properties:
   a) \( B \) is an upper bound for \( S \).
   b) No number less than \( B \) is an upper bound for \( S \).

   (b) State the least upper bound axiom (with all the appropriate conditions).

   Every nonempty set \( S \) of real numbers that is bounded above has a supremum.

3. (20 pts) Show that for any positive integer \( n \) we have

   \[
   1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}
   \]

   Hint: use induction.

   Base case: \( n = 1 \). We have \( 1 \leq 2 \).

   Induction step:

   Assume \( 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} \), want to show \( 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \).

   We know by the induction hypothesis that \( 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \).

   Now we need to check that \( 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \).
Since everything is positive we can square both sides and this is equivalent to checking

\[ 4 + \frac{4\sqrt{n}}{\sqrt{n + 1}} + \frac{1}{n + 1} \leq 4(n + 1). \]

The left hand side yields

\[ 4 + \frac{4\sqrt{n}}{\sqrt{n + 1}} + \frac{1}{n + 1} = \frac{4n + 4 + 4\sqrt{n^2 + n}}{n + 1}. \]

So we check that \(4n + 4 + 4\sqrt{n^2 + n} \leq 4(n + 1)^2\).

We know that \(n^2 + n \leq n^2 + 2n + 1\), thus

\[ 4n + 4 + 4\sqrt{n^2 + n} \leq 4n^2 + 4n + 4(n + 1) = 4(n + 1)^2. \]

\[ \square \]

4. (25 pts) Let \(f\) and \(g\) be step functions defined the following way:

\[ f(x) = \begin{cases} 1 & \text{if } 2 \leq x < 3, \\ 2 & \text{if } 3 \leq x < 5, \\ 3 & \text{if } 5 \leq x \leq 8, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{if } 1 \leq x < 4, \\ -1 & \text{if } 4 \leq x < 6. \end{cases} \]

(a) Give a short definition of a step function.

A function \(s\), whose domain is a closed interval \([a, b]\), is called a step function if there is a partition \(P = \{x_0, x_1, \ldots, x_n\}\) of \([a, b]\) such that \(s\) is constant on each open subinterval of \(P\). That is to say, for each \(k = 1, 2, \ldots, n\), there is a real number \(s_k\) such that \(s(x) = s_k\) if \(x_{k-1} < x < x_k\).

Note: the textbook insists on using closed intervals in the definition, but this is not taken that seriously in practice.

(b) Show that \(f + g\) is a step function and describe it fully! (Be careful with the domain!)

Note that \(f\) is defined on \([2, 8]\) and \(g\) is defined on \([1, 6]\). So \(f + g\) can only be defined on \([2, 6]\). Thus strictly speaking as a step function \(f + g\) is only defined on \([2, 6]\). (If we take the strict definition with the closed intervals then this wouldn’t even be a step-function.) We have

\[ f + g = \begin{cases} 3 & x \in [2, 3) \\ 4 & x \in [3, 4) \\ 1 & x \in [4, 5) \\ 2 & x \in [5, 6) \end{cases} \]

(c) Evaluate the integral \(\int_2^5 (f(x) + g(x))\,dx\).

Using part b) we get \(\int_2^5 (f(x) + g(x))\,dx = 3 + 4 + 1 = 8\).
5. (20 pts)

(a) Recall that the conjugate of a complex number \( z = a + bi \) (with \( a, b \in \mathbb{R} \)) is defined as \( \bar{z} = a - bi \). Prove that for any two complex numbers \( z_1, z_2 \) we have \( \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \) and \( \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \).

Let \( z_1 = a_1 + b_1i, \ z_2 = a_2 + b_2i \). Then

\[
z_1 + z_2 = a_1 + b_1i + a_2 + b_2i = (a_1 + a_2) + i(b_1 + b_2).
\]

So \( \overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2) \).

Also \( \bar{z}_1 = a_1 - b_1i \) and \( \bar{z}_2 = a_2 - b_2i \). Thus

\[
\bar{z}_1 + \bar{z}_2 = (a_1 + a_2) - i(b_1 + b_2) = \overline{z_1 + z_2}.
\]

Similarly we have

\[
z_1 \cdot z_2 = a_1a_2 + a_1b_1i + a_2b_1i - b_1b_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).
\]

So \( \overline{z_1 \cdot z_2} = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1) \).

And we have \( \bar{z}_1 \cdot \bar{z}_2 = (a_1 - b_1i)(a_2 - b_2i) = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1) \).

Thus \( \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \).

(b) Show that if the complex number \( z \) is a solution of \( z^6 + 3z^5 - 4z^4 + 8z^2 - z + 1 = 0 \) then \( \bar{z} \) is also a solution.

Hint: You cannot solve the equation explicitly. You have to use the properties of the \( z \to \bar{z} \) function to solve the problem. (If you are doing complicated computations then you are not on the right track...)

This means we must show \( \bar{z}^6 + 3\bar{z}^5 - 4\bar{z}^4 + 8\bar{z}^2 - \bar{z} + 1 = 0 \) Thus it suffices to show that

\[
\bar{z}^6 + 3\bar{z}^5 - 4\bar{z}^4 + 8\bar{z}^2 - \bar{z} + 1 = \bar{z}^6 + 3\bar{z}^5 - 4\bar{z}^4 + 8\bar{z}^2 - \bar{z} + 1
\]

as \( 0 = 0 \). However this follows from repeated application of part a) because we know that \( \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \) and \( \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \). Also from class we know that for \( a \in \mathbb{R} \) and \( z \in \mathbb{C} \) we have that \( \bar{a} = a \) and \( \overline{a \cdot z} = a \bar{z} \).