Homework 2

Due: February 16, 2010, beginning of the class. Late homework will not be accepted.

1. (Exercise 2.2) Assume that a sequence of independent events \( \{A_i\} \) satisfy \( \sum_{n=1}^{\infty} P(A_i) = \infty \). Let

\[ \tau_k = \min\{n : \sum_{i=1}^{n} 1_{A_i} = k\} \]

By the (second) Borel-Cantelli lemma with probability one we will have infinitely many of the \( A_i \)'s occurring, i.e. \( \sum_{k=1}^{\infty} 1_{A_k} = \infty \) and \( P(\tau_k < \infty) = 1 \) for all \( k \). Prove the slightly stronger statement

\[ k = E \sum_{i=1}^{\tau_k} P(A_i). \]

Why is this a stronger statement?

Hint: construct a martingale using the random variables \( 1_{A_i} \) and use the fact that \( \tau_k \) is a stopping time.

2. In each of the following cases check if the process is a standard Brownian motion.

   (a) \( X_t = \frac{1}{\sqrt{t}} B_{t^2} \) where \( B_t \) is a standard BM.

   (b) \( Y_t = \sin(\alpha) B_t^{(1)} + \cos(\alpha) B_t^{(2)} \) where \( B_t^{(1)} \) and \( B_t^{(2)} \) are independent standard BM's and \( \alpha \in \mathbb{R} \).

   (c) \( Z_t = \begin{cases} B_t & \text{if } 0 \leq t \leq 1 \\ B_{t+1} - B_2 + B_1 & \text{if } t \geq 1 \end{cases} \)

   where \( B_t \) is a standard BM.

3. (Exercise 3.1 (b)-(d)) Let \( U_t \) be a standard Brownian bridge (see page 41 for the definition).

   (a) Show that \( \text{Cov}(U_s, U_t) = s(1-t) \) for \( 0 \leq s \leq t \leq 1 \).

   (b) Let \( X_t = g(t)B_{h(t)} \), and find functions \( g \) and \( h \) such that \( X_t \) has the same covariance as the Brownian bridge. (\( B_t \) is a standard BM.)

   (c) Show that \( Y_t = (1+t)U_{t/(1+t)} \) is a BM on \([0, \infty)\).

Hint: (c) should help with (b)...

4. An urn contains \( a \) red and \( b \) black balls. In each step we draw a ball randomly and replace it with two balls of the same color. (Essentially in each step we add a new ball to the urn whose color is determined randomly.) Let \( X_n \) be the ratio of red balls in the urn after the \( n^{th} \) step. \( X_0 = a/(a+b) \).

Show that \( X_n \) is a martingale and it converges almost surely.

**Bonus problem.** Assume that \( X \) and \( Y \) are independent, identically distributed with mean 0 and variance 1. Show that if the random variables \( X + Y, X - Y \) are independent then \( X \) and \( Y \) are standard normals.