Due: March 16, 2010, beginning of the class. Late homework will not be accepted.

1. (Exercise 6.1) Use the Itô isometry to calculate the variances of
\[ \int_0^t |B_s|^{1/2} dB_s \quad \text{and} \quad \int_0^t (B_s + s)^2 dB_s \]

2. (Exercise 6.2) The integrals
\[ I_1 = \int_0^t B_s ds \quad \text{and} \quad I_2 = \int_0^t B_s^2 ds \]
are not stochastic integrals although they are random variables. In these cases we just integrate a (random) continuous function the usual (traditional) way. Find the mean and variance of \( I_1 \) and \( I_2 \).
Also: compute the moment generating function of \( I_1 \). (It’s easy once you have the mean and variance...)

3. Prove the following (weaker) version of the iterated logarithm theorem:
\[ \limsup_{t \to \infty} \frac{B^*_t}{\sqrt{2t \log \log t}} \leq 1 \ \text{a.s.} \]
where \( B^*_t = \max_{0 \leq s \leq t} B_s \).

Hint: Let \( t_n = c^n \) with \( c > 1 \) and show that \( B^*_t \geq c\sqrt{2t_n \log \log t_n} \) a.s. if \( n \) large enough.

4. Use the definition of the Itô integral to prove that
\[ \int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds. \]

Bonus problem.

(a) Let \( X_t \) be a continuous non-negative martingale with \( X_0 = 1 \) and \( X_t \to 0 \) a.s. Show that for any \( x \geq 1 \)
\[ P(\sup_{t \geq 0} X_t \geq x) = 1/x. \]

Hint: consider the hitting time \( \tau_x \) of \( x \) and the stopped martingale \( X_{t \wedge \tau_x} \).

(b) Use the previous result to show that if \( b > 0 \) then \( Y = \sup_t (B_t - bt) \) is an exponential random variable with parameter \( 2b \). (I.e. \( P(Y \leq x) = 1 - e^{-2bx} \) if \( x \geq 0 \).)