Math/Stat 831 – Fall 2008

# Final Exam

1. **(9 points)** Let $X_1, X_2, \ldots$ be i.i.d. with mean 0 and variance 1. Let $a_n$ be an increasing sequence of positive numbers, and denote $A_n = (\sum_{k=1}^n a_n^2)^{1/2}$. Show that if $a_n/A_n \to 0$ then

$$\frac{\sum_{k=1}^{n} a_k X_k}{A_n} \Rightarrow N(0,1),$$

i.e. the normalized sums converge to a standard normal random variable in distribution.

2. **(10 points)** Suppose that for two random variables $X$ and $Y$ the following statements hold:

- $X - Y$ and $Y$ are independent,
- $X - Y$ and $X$ are independent.

(a) Show that if the characteristic functions of $X$ and $X-Y$ are denoted by $\psi_X(t)$ and $\psi_{X-Y}(t)$ then

$$\psi_X(t)\,(1 - |\psi_{X-Y}(t)|^2) = 0. \qquad (1)$$

(b) Show that if equation (1) holds for all $t$ then $|\psi_{X-Y}(t)| = 1$ if $t$ is close enough to 0.

(c) Show that $X - Y$ is constant with probability 1.

3. **(10 points)** Let $\xi_n$ be independent random variables with distribution

$$\xi_n = \begin{cases} 1 & \text{with probability } (2n)^{-1} \\ 0 & \text{with probability } 1 - 1/n \\ -1 & \text{with probability } (2n)^{-1} \end{cases}$$

Let $X_0 = 0$ and define $X_n$ recursively by

$$X_{n+1} = |\xi_n| n X_n + \xi_n \mathbf{1}(X_n = 0).$$

(a) Show that $X_n$ is a martingale.

(b) Show that $X_n \xrightarrow{\mathbf{P}} 0$. (Hint: it's easy to estimate $\mathbf{P}(X_n = 0)\ldots$)

(c) Show that $X_n$ does not converge to 0 a.s. (Hint: then $\xi_n$ should be 0 eventually.)

4. **(9 points)** Consider the Lebesgue probability space on the interval $[0,1)$. (I.e. the state space is $\Omega = [0,1)$, the $\sigma$-field is the set of Lebesgue measurable sets and the measure is the Lebesgue measure.) We define the random variable $X$ as

$$X(\omega) = \begin{cases} 2\omega, & \text{if } 0 \le \omega < 1/2, \\ 2\omega - 1 & \text{if } 1/2 \le \omega < 1. \end{cases}$$

Compute the conditional expectation $\mathbf{E}(Y|X)$ where $Y : [0,1) \to \mathbf{R}$ is a measurable function.

5. **(12 points)** Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbf{P}(X_n = 1) = p, \mathbf{P}(X_n = -1) = 1 - p$ where $1/2 < p < 1$ is a fixed constant. Let $S_n = \sum_{k=1}^n X_n$ (with $S_0 = 0$) and $\varphi(x) = (1/p - 1)^x$.

(a) Show that $\varphi(S_n)$ is a martingale with respect to the natural filtration.

(b) For given $a, b$ positive integers consider the first time $S_n$ reaches $a$ or $-b$:

$$\tau = \inf\{k : S_k = a \quad \text{or} \quad S_k = -b\}.$$

Show that $\tau$ is a stopping time with respect to the natural filtration.

(c) Show that $\mathbf{E}\varphi(S_\tau) = 1$ and compute the probability of reaching $a$ before $b$, i.e. $\mathbf{P}(S_\tau = a)$.

(d) Find a suitable martingale to compute $\mathbf{E}\tau$. (Hint: what is the expectation of $X_n$?)

**Please present your solutions in a clear manner. Show all your work.**