Due: September 16, 2010, beginning of the class. Late homework will not be accepted.

1. In each of the following cases construct a probability space to model the corresponding random experiment. Be sure to describe each component in \((\Omega, \mathcal{F}, P)\) carefully. (There might be several possible correct solutions.)

   (a) We flip three fair coins and throw two dice.
   (b) We throw a fair die and if it shows the number \(n\) then we flip a coin \(n\) times.

2. Let \(\{\mathcal{F}_\alpha, \alpha \in A\}\) be a (not necessarily countable) collection of \(\sigma\)-fields on the sample space \(\Omega\).

   (a) Show that \(\bigcap_{\alpha \in A} \mathcal{F}_\alpha\) is a \(\sigma\)-field.
   (b) Show that for any collection of subsets \(G\) in \(\Omega\) there is a smallest \(\sigma\)-field containing \(G\). (This is the \(\sigma\)-field generated by \(G\): \(\sigma(G)\).)

3. Let \((\Omega, \mathcal{F}, P)\) be a probability space. Show that for every \(A, B, C \in \mathcal{F}\)

   \[
P(A \circ B) \leq P(B \circ C) + P(A \circ C)
   \]

   \(A \circ B\) denotes the symmetric difference: \((A^c \cap B) \cup (A \cap B^c)\).

4. We have a large empty urn and infinitely many balls numbered with the positive integers. At \(t = 0\) we add the balls numbered with \(1, 2, \ldots, 10\) into the urn then choose one randomly and throw it away. At \(t = 1/2\) we add the balls \(11, 12, \ldots, 20\) into the urn then choose one randomly (out of the 19) and throw it away. We repeat this infinitely many times: at time \(t = 1 - 1/2^n\) we add the balls \(10n + 1, 10n + 2, \ldots, 10(n + 1)\) into the urn, choose one randomly and we throw it away. Show that with probability one at time \(t = 1\) the urn will be empty.

   Hint: Show that for any \(k \in \mathbb{Z}_+\) the probability that the ball \(k\) will be in the urn at time \(t = 1\) is zero.

5. Let \(X\) be a uniform random variable on \([0, 1]\) and consider \(Y = X^2\). Describe

   (a) the distribution of \(Y\),
   (b) the distribution function of \(Y\).

**Bonus problem.** The following statement shows that if you want to prove an identity or inequality relating probabilities of certain events – like in problem 2 – then it is enough to check it on the trivial probability space (i.e. where \(\mathcal{F} = \{\emptyset, \Omega\}\)).

Suppose that \(A_1, A_2, \ldots, A_n \in \mathcal{F}\) and \(B_1, B_2, \ldots, B_k \in \sigma(A_i : i = 1 \ldots n)\). (This means that each \(B_j\) may be expressed from the \(A_i\)’s using the usual set operations.) Let \(c_1, c_2, \ldots, c_k\) be real numbers. Then

\[
\sum_{j=1}^{k} c_j P(B_j) \geq 0
\]

holds for all probability spaces if and only if it holds for the trivial probability space. (The same statement holds with \(=\) instead of \(\geq\).)

Hint: Try to find 'building blocks’ for the \(\sigma\)-field generated by the \(A_i\)’s.