Math/Stat 831 – Fall 2010

Homework 2

Due: September 30, 2010

1. Let X be a an absolutely continuous random variable (wrt Lebesgue) and suppose that g is a measurable function for which we have $g \geq 0$ or $Eg(X)| < \infty$. Give a full proof of the identity $Eg(X) = \int f(x)g(x)dx$.

2. (a) Show that if X and Y are independent with distributions μ and ν then

$$P(X+Y=0) = \sum_y \mu(\{-y\})\nu(\{y\})$$

Hint: you first need to show that apart from countably many values y we have $\mu(\{-y\})\nu(\{y\}) = 0$ which shows that the sum always makes sense.

(b) If we also know that X has continuous distribution (i.e. its distribution function is continuous) then

$$P(X=Y)=0$$

3. Suppose that X and Y are independent random variables with the following distribution:

$$P(X=k) = P(Y=k) = p(1-p)^k, \qquad k \in \mathbb{Z}^+ = \{0,1,2,\dots\}$$

where $0 < p < 1$ is fixed. (I.e. they have geometric distribution with parameter p.)

Is it true that the random variables $A = \min(X,Y)$ and $B = \max(X,Y) - \min(X,Y)$ are independent? What is the distribution of B?

4. Show that if $X \geq 0$ is an integer valued random variable than

$$EX = \sum_{k \geq 1} P(X \geq k).$$

Find a similar expression for EX^2. (It should have the form of $\sum_{k\geq 1} \varphi(k) P(X \geq k)$.)

5. Let $f : [0,1] \to \mathbb{R}$ be a bounded, three time continuously differentiable function. Evaluate the following limit:

$$\lim_{n\to\infty} n \int_0^1 \int_0^1 \dots \int_0^1 \left[f\left(\frac{1}{n}(x_1 + x_2 + \dots + x_n)\right) - f(1/2)\right] dx_1\, dx_2 \dots dx_n$$

Hint: use Taylor expansion and the Weak Law of Large Numbers.

Bonus problem. Consider the probability space $([0,1], \mathcal{B}, \lambda)$ (where λ is the Lebesgue measure) and let F be a distribution function. Construct an infinite sequence of i.i.d. random variables $X_1, X_2 \dots$ with common distribution function F.
Note: Kolmogorov's extension theorem shows that we can define this i.i.d. sequence on an infinite product space. This problem provides an explicit construction on a simple probability space.