1. Let $X$ be an absolutely continuous random variable (wrt Lebesgue) and suppose that $g$ is a measurable function for which we have $g \geq 0$ or $Eg(X) < \infty$. Give a full proof of the identity $Eg(X) = \int f(x)g(x)dx$.

2. (a) Show that if $X$ and $Y$ are independent with distributions $\mu$ and $\nu$ then
   
   $$P(X + Y = 0) = \sum_y \mu(\{-y\})\nu(\{y\})$$
   
   Hint: you first need to show that apart from countably many values $y$ we have $\mu(\{-y\})\nu(\{y\}) = 0$ which shows that the sum always makes sense.

   (b) If we also know that $X$ has continuous distribution (i.e. its distribution function is continuous) then
   
   $$P(X = Y) = 0$$

3. Suppose that $X$ and $Y$ are independent random variables with the following distribution:
   
   $$P(X = k) = P(Y = k) = p(1 - p)^k, \quad k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$$
   
   where $0 < p < 1$ is fixed. (I.e. they have geometric distribution with parameter $p$.) Is it true that the random variables $A = \min(X, Y)$ and $B = \max(X, Y) - \min(X, Y)$ are independent? What is the distribution of $B$?

4. Show that if $X \geq 0$ is an integer valued random variable than
   
   $$EX = \sum_{k \geq 1} P(X \geq k).$$
   
   Find a similar expression for $EX^2$. (It should have the form of $\sum_{k \geq 1} \varphi(k)P(X \geq k)$.)

5. Let $f : [0, 1] \to \mathbb{R}$ be a bounded, three time continuously differentiable function. Evaluate the following limit:
   
   $$\lim_{n \to \infty} n \int_0^1 \int_0^1 \ldots \int_0^1 \left[f \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) - f(1/2)\right] dx_1 dx_2 \ldots dx_n$$
   
   Hint: use Taylor expansion and the Weak Law of Large Numbers.

**Bonus problem.** Consider the probability space $([0, 1], \mathcal{B}, \lambda)$ (where $\lambda$ is the Lebesgue measure) and let $F$ be a distribution function. Construct an infinite sequence of i.i.d. random variables $X_1, X_2 \ldots$ with common distribution function $F$.

Note: Kolmogorov’s extension theorem shows that we can define this i.i.d. sequence on an infinite product space. This problem provides an explicit construction on a simple probability space.