Math/Stat 831 – Fall 2010

Homework 4

Due: October 28, 2010

1. Show that the difference of two identically distributed independent random variables cannot have a uniform distribution on [-1,1].

2. Show that if $X_n \xrightarrow{P} X$ then $X_n \Rightarrow X$.

3. Prove that if X_n converges in distribution to a constant c then $X_n \xrightarrow{P} c$.

4. Let $X_1, X_2, \ldots$ i.i.d with distribution function $F(x)$. Denote the maximum of the first n element by M_n. Show that if

$$\lim_{x \to \infty} x^\alpha (1 - F(x)) = b$$

with fixed positive constants α, b then $n^{-1/\alpha} M_n$ converges in distribution and identify the limiting distribution.
Hint: you can use the convergence of the distribution functions to prove the weak limit.

5. Use characteristic functions to prove the following identity

$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right).$$

6. Let $X_1, X_2, \ldots$ independent with the following distribution:

$$\mathbf{P}(X_m = m) = \mathbf{P}(X_m = -m) = \frac{1}{2m^2}, \quad \mathbf{P}(X_m = 1) = \mathbf{P}(X_m = -1) = \frac{1}{2} - \frac{1}{2m^2}.$$

Let $S_n = X_1 + \cdots + X_n$. Show that $\frac{Var(S_n)}{n} \to 2$, but $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$.

Bonus problem
Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli(1/2) random variables. Let ν_n denote the index k when we first have at least n zeros and n ones among $X_1, \ldots, X_k$. Prove that $\frac{\nu_n - 2n}{\sqrt{n}}$ converges in distribution and find the limit.