1. Assume that $\mu_n \Rightarrow \mu$ and $\int x^k d\mu_n \to m_k < \infty$ for all $k \geq 1$. Show that for every $k > 0$ the $k^{th}$ moment of $\mu$ is finite and it is given by $m_k$.
(We basically proved this in class, but you should provide a detailed proof.)

2. Let $X_1, X_2, \ldots$ be independent random variables and $S_n = X_1 + \cdots + X_n$. Assume that almost surely $|X_i| \leq M$ for all $i \geq 1$ with a given constant $M < \infty$. Show that if $\text{Var} S_n \to \infty$ then $\frac{S_n - ED_n}{\sqrt{\text{Var} S_n}} \Rightarrow N(0, 1)$.

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function $F$ and a continuous density $f$. We know that $F_n(x) := \frac{1}{n} \sum_{k=1}^{n} 1(X_k \leq x)$ converges uniformly to $F(x)$ with probability one. Now we will look at the empirical distribution on a finer scale. Let $c \in \mathbb{R}$ be a number with $f(c) > 0$ and consider $N_n(a, b) = \sum_{k=1}^{n} 1(X_k \in (c + \frac{a}{n}, c + \frac{b}{n}))$. Show that $N_n(a, b)$ converges in distribution for any $a < b$ and find the limit.

4. Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Find a deterministic sequence $a_n$ so that $Y_n = a_n \frac{X_1}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}}$

converges weakly to a non-constant distribution and identify the limit.

**Bonus problem**
Let $X$ and $Y$ be identically distributed random variables with mean zero and variance one. Assuming that $X - Y$ and $X + Y$ are independent, find the distribution of $X$. 