Math/Stat 831 – Fall 2010

Homework 6

Due: December 9, 2010

Note: do not leave this assignment in my mailbox. Submit it in class or email me the electronic version.

1. Let X and Y be real valued random variables and assume that $X \in \sigma(Y)$. Show that there exists a Borel measurable $g : \mathbb{R} \to \mathbb{R}$ so that $X = g(Y)$.
   Hint: first try the case when X is a simple function.

2. Assume that X, Y are i.i.d. with $E|X| < \infty$. Show that $E(X|X + Y) = \frac{X+Y}{2}$.

3. Let ν and τ be stopping times with respect to the same filtration $\{\mathcal{F}_n\}$. Show that $\nu \wedge \tau$ and $\nu + \tau$ are also stopping times with respect to $\{\mathcal{F}_n\}$.

4. Let S_n denote the position of a simple random walk after n steps. (I.e. $S_n = X_1 + \cdots + X_n$ with X_i i.i.d. $P(X_i = \pm 1) = 1/2$.) For which polynomials $g(x)$ will $g(S_n)$ be a martingale?

5. For a parameter $a \in (0, 1)$ let $p_a(x_1, x_2, \ldots, x_n)$ denote the probability that the first n terms in the i.i.d. sequence $X_1, X_2, \ldots$ of Bernoulli(a) random variables are exactly $x_1, x_2, \ldots, x_n$. Let $a \neq b$ and
$$Z_n = \frac{p_b(X_1, X_2, \ldots, X_n)}{p_a(X_1, X_2, \ldots, X_n)}$$
   where $X_1, X_2, \ldots$ are i.i.d. Bernoulli(c) random variables. Prove that Z_n is a martingale if and only if $c = a$.

6. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_i = 1$ and $P(Y_i = 1) < 1$. Show that $X_n = Y_1 Y_2 \cdots Y_n$ is a martingale and that $X_n \to 0$ a.s.

Bonus problem

Consider a simple random walk on the plane (i.e. each step is an independent one unit jump to one of the four directions with equal probability) and denote the distance from the origin by S_n. Let $\nu_r = \inf\{S_n > r\}$. Show that $S_n^2 - n$ is a martingale and that $r^{-2} E\nu_2 \to 1$ as $r \to \infty$.