Math/Stat 831 – Fall 2010

Midterm Exam

Instructions: You may not work together with any other person while solving the problems of this exam. You may use your textbook and your notes, but you may not use outside sources. You do not need to reprove anything covered in class, but be sure that you properly identify the theorem or lemma you use.

The solutions to this exam are due by **4 PM on Wednesday, November 3**. *Late exams will not be accepted or graded.* You may submit your solutions in writing (just drop it into my mailbox on the second floor of Van Vleck) or electronically.

1. Let $X_1, X_2, \dots$ be i.i.d. standard normal random variables.

   (a) Prove that
   $$P(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1) = 1.$$
   You may use the following bounds on the tail of the standard normal distribution for $x > 0$ (which follow easily by integration):
   $$\left(\frac{1}{x} - \frac{1}{x^3}\right)\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \le P(X_i \ge x) \le \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

   (b) Let $S_n = X_1 + \cdots + X_n$. Show that
   $$P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} \le 1) = 1.$$

   Note: this is a weaker (and lot easier) version of the Iterated Logarithm Theorem which states that that
   $$P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1) = 1.$$

2. Let X be a Cauchy distributed random variable and denote by $\{x\}$ the fractional part of $x \in \mathbb{R}$. Show that $Y_n = \{nX\}$ converges in distribution and find the limit.

   Note: the density of the Cauchy distribution is $\frac{1}{\pi(1+x^2)}$ and $\{x\} = x - [x]$ where $[x]$ is the largest integer at most as big as x. You should try to imagine what the distribution of Y_n looks like before you start computing.

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Evaluate
   $$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdot \cdots \cdot x_n)^{1/n}\right) dx_1\, dx_2 \dots dx_n$$
   Hint: use the law of large numbers. You might want to try $f(x) = x$ first.