Math/Stat 832 – Spring 2011

Homework 4

Due: April 7, 2011

1. The baker’s map is a \([0, 1)^2 \to [0, 1)^2\) function given by

\[
\phi(u, v) = \begin{cases} 
(\{2u\}, v/2) & \text{if } u < 1/2 \\
(\{2u\}, v/2 + 1/2) & \text{if } u \geq 1/2
\end{cases}
\]

where \(\{x\}\) denotes the fractional part.

Show that this map is invariant with respect to the Lebesgue measure \(\lambda\) on \([0, 1)^2\) and for any \(f : [0, 1)^2 \to \mathbb{R}\) the sequence \((f(\omega), f(\phi(\omega)), \ldots, f(\phi^k \omega), \ldots)\) is stationary and ergodic on the probability space \(([0, 1)^2, B^2, \lambda)\).

2. Give an example of an ergodic measure preserving transformation \(T : \Omega \to \Omega\) and probability space \((\Omega, \mathcal{F}, P)\) for which \(T^2\) is not ergodic.

3. Let \(\varphi(x) = \{\frac{1}{x}\}\) for \(x \in (0, 1)\). Show that \(\varphi\) preserves the measure

\[
\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1 + x}, \quad A \subset (0, 1).
\]

4. Let \(X_0, X_1, \ldots\) be a stationary sequence and denote the distribution of \(X_0\) by \(\mu\). Show that the sequence is ergodic if and only if

\[
\lim_{t \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k)g(X_0) = Ef(X_0)Eg(X_0)
\]

for every \(f, g \in L^2(\mu)\).

**Bonus problem:** Let \(\theta\) be irrational. Show that the map \(T : (x, y) \to (x + \theta, x + y)\) is invariant for the Lebesgue measure on the unit square (the addition is meant mod 1 here) and that it is ergodic.