Lecture 1: Basic random matrix models

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Our aim in this course to study the asymptotic behavior of the spectrum of certain random matrices.

Wigner Matrices

**Definition 1** (real Wigner matrices). For $1 \leq i < j < \infty$ let $X_{i,j}$ be i.i.d. (real) random variables with mean 0 and variance 1 and set $X_{j,i} = X_{i,j}$. Let $X_{i,i}$ be i.i.d. (real) random variables (with possibly a different distribution) with mean 0 and variance 1. Then $M_n = \left[ X_{i,j} \right]_{i,j=1}^n$ will be a random $n \times n$ symmetric matrix.

**Definition 2** (complex Wigner matrices). For $1 \leq i < j < \infty$ let $X_{i,j}$ be i.i.d. (complex) random variables with mean 0, $E|X_{i,j}|^2 = 1$ and set $X_{j,i} = \overline{X_{i,j}}$. Let $X_{i,i}$ be i.i.d. (real) random variables with mean 0 and variance 1. Then $M_n = \left[ X_{i,j} \right]_{i,j=1}^n$ will be a random $n \times n$ hermitian matrix.

In both cases there are $n$ random eigenvalues which we will denote by $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$.

(We will denote the dependence on $n$). Fact (which we will prove later): these are continuous functions of $M_n$ hence they are random variables themselves.

We would like to study the scaling limit of the empirical spectral measure

$$\nu_n^* = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}.$$ 

This is a random discrete probability measure which puts $n^{-1}$ mass to each (random) eigenvalue. The following picture shows the histogram of eigenvalues for a certain $200 \times 200$ Wigner matrix.
The picture suggests that there is a nice deterministic limiting behavior. In order to figure out the right scaling, we first compute the order of the empirical mean and second moment of the eigenvalues.

$$\frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} \text{Tr} M_n = \frac{1}{n} \sum_{i=1}^{n} X_{i,i}$$

$$\frac{1}{n} \sum_{i=1}^{n} \lambda_i^2 = \frac{1}{n} \text{Tr} M_n^2 = \frac{1}{n} \sum_{i,j=1}^{n} X_{i,j}^2$$

The first moment converges to 0 by the strong law of large numbers. However the second moment is of $O(n)$ as we have about $n^2/2$ independent terms in the sum with a normalization of $\frac{1}{n}$ instead of $\frac{1}{n^2}$. This suggests that in order to see a meaningful limit, we need to scale the eigenvalues (or the matrix) by $\frac{1}{\sqrt{n}}$.

The following theorem states that in case we indeed have a deterministic limit.

**Theorem 3** (Wigner’s semicircle law). Let

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}}.$$

be the normalized empirical spectral measure. Then as $n \to \infty$ we have

$$\nu_n \Rightarrow \nu \quad \text{a.s.}$$

where $\nu$ has density

$$\frac{d\nu}{dx} = \frac{1}{2\pi} \sqrt{4 - x^2} 1\{|x| \leq 2\}.$$

(There will be some assumptions on the distribution of the random entries of $M_n$.)

**Gaussian Ensembles**

We also discussed some special Wigner matrix models.

**Definition 4** (GOE). Consider a real Wigner matrix where $X_{i,j} \sim N(0,1)$ and $X_{i,i} \sim \sqrt{2}N(0,1)$. The resulting random matrix model is called *Gaussian Orthogonal Ensemble* (or GOE).

Another construction: let $a_{i,j}, i, j \in \mathbb{Z}$ be i.i.d. standard normals and $A_n = [a_{i,j}]_{i,j=1}^{n}$. (Note that this is not a symmetric matrix!). Then the distribution of $M_n = \frac{A_n + A_n^T}{\sqrt{2}}$ is GOE.

It is easy to check the following useful fact: if $C \in \mathbb{R}^{n \times n}$ is orthogonal (i.e. $C C^T = I$) the $C^T M_n C$ has the same distribution as $M_n$. (The GOE is invariant to orthogonal conjugation.) It is a bit harder (we will prove it later) that one can actually compute the joint eigenvalue density which is given by

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_1} \prod_{i<j} |\lambda_j - \lambda_i| e^{-\frac{1}{2} \sum_{i=1}^{n} \lambda_i^2}.$$

Here $Z_1$ is an explicitly computable normalizing constant (which also depends on $n$).
Definition 5 (GUE). Consider a complex Wigner matrix where \( X_{i,j} \) is standard complex Gaussian (i.e. \( X_{i,j} \sim N(0, \frac{1}{2}) + iN(0, \frac{1}{2}) \)) and \( X_{i,i} \sim N(0, 1) \) (real). The resulting random hermitian matrix model is called Gaussian Unitary Ensemble (or GUE).

Another construction: let \( a_{i,j}, i,j \in \mathbb{Z} \) be i.i.d. standard complex Gaussians and \( A_n = [a_{i,j}]_{i,j=1}^n \). (Note that this is not a symmetric matrix!). Then the distribution of \( M_n = \frac{A_n + A_n^T}{\sqrt{2}} \) is GUE.

As the name suggests, GUE is invariant under unitary conjugation. If \( C \in \mathbb{C}^{n \times n} \) is unitary (i.e. \( CC^* = I \)) the \( C^TM_mC \) has the same distribution as \( M_n \). (We will later show that the joint eigenvalue density is given by

\[
f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_2} \prod_{i<j} |\lambda_j - \lambda_i|^2 e^{-\frac{1}{2} \sum_{i=1}^n \lambda_i^2}.
\]

Here \( Z_2 \) is an explicitly computable normalizing constant (which also depends on \( n \)).

One can see the similarity between the two densities: they are contained in the following one-parameter family of densities:

\[
f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_\beta} \prod_{i<j} |\lambda_j - \lambda_i|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2}.
\]

(1)

For a given \( \beta > 0 \) the resulting distribution (on ordered \( n \)-tuples in \( \mathbb{R} \)) is called Dyson’s \( \beta \)-ensemble. For \( \beta = 1 \) one gets the eigenvalue density of GOE, for \( \beta = 2 \) we get the GUE. The \( \beta = 4 \) case is also special: it is related another classical random matrix model, the Gaussian Symplectic Ensemble (GSE), which can be defined using quaternions.

For other values of \( \beta \) there are no ’nice’ random matrices in the background. (We will see that one can still build random matrices from which we get the general \( \beta \)-ensemble, but they won’t have such nice symmetry properties.)

Later in the semester we will show that if one scales the \( \beta \) ensembles properly (’zooming in’ to see the individual eigenvalues near a point) then one gets a point process limit. The limiting point process is especially nice in the \( \beta = 2 \) case (GUE). It is conjectured that its distribution appears among the critical line zeros of the Riemann-\( \zeta \) function.

Another symmetric random matrix model

Another way of constructing a symmetric random matrix is the following. Let \( a_{i,j} \) be i.i.d random real random variables with mean 0 and variance 1. Let \( A = [a_{i,j}]_{i,j=1}^n \in \mathbb{R}^{n \times m} \) be a random matrix with \( n \) rows and \( m \) columns (with \( n \leq m \)). Then \( M_n = AA^T \) is a (positive) symmetric random matrix. We will show that the appropriately normalized empirical spectral measure will converge to a deterministic limit. (This is the Marchenko-Pastur law). A similar statement holds if we construct our matrix from i.i.d. complex random variables.