Lectures 20 – 22: Scaling limit of $\beta$-ensembles

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1 Edelman-Sutton Conjecture (cont.)

1.1 Existence of eigenvalue-eigenfunction pair for the smallest eigenvalue of $cH_{\beta}$

In the last lecture, we proved $\exists \{f_n\}$ and a nonnegative random variable $K$ with $\|f_n\|_2^2 = 1$, $\|f_n\|_2^2 (K \text{ a.s. } \forall n)$, so that

$$\langle f_n, H_{\beta} f_n \rangle \to \tilde{\Lambda}_0 = \inf_{\|f\|_2^2 = 1} \langle f, H_{\beta} f \rangle \text{ a.s.}$$

Also, we proved the lemma: If $\{f_n\}$ is bounded in $L^\ast$, then we can find a subsequence $\{f_{n_k}\}$ so that

1) $f_{n_k} \to \tilde{f}$ in $L^2$
2) $f_{n_k}' \to \tilde{f}'$ weak convergence in $L^2$
3) $f_{n_k} \to \tilde{f}$ uniformly on compacts
4) $f_{n_k} \to \tilde{f}$ weakly in $L^\ast$

Without loss of generality, we may assume $\{f_n\}$ itself has the above properties.

We will show that $\tilde{\Lambda}_0 = \langle \tilde{f}, H_{\beta} \tilde{f} \rangle$ a.s. and that $(\tilde{\Lambda}_0, \tilde{f})$ is an eigenvalue-eigenfunction pair with the smallest eigenvalue.

Proof. In the previous lecture, we proved $\exists$ a nonnegative random variable $C(B)$:

$$\sup_x \frac{\max(|\bar{B}'(x)|, |B(x) - \bar{B}(x)|)}{\sqrt{x \log(2 + x)}} < C(B) < \infty.$$ 

Hence, $\forall \epsilon > 0$, $\exists$ a random variable $X$, so that $|\bar{B}'(x)|/\epsilon(1 + x)$ and $|B(x) - \bar{B}(x)|/\epsilon \sqrt{1 + x}$, $\forall x > X$. Then, $\forall f_n$,

$$\langle f_n, H_{\beta} f_n \rangle = \int_0^\infty \left( |f_n'|^2 + x f_n^2 \right) dx + \frac{2}{\sqrt{\beta}} \int_0^x f_n^2 \bar{B}' dx + \frac{4}{\sqrt{\beta}} \int_0^x f_n f_n' (\bar{B} - B) dx + E \quad (1)$$

1
where \(|E| \leq \epsilon C(\beta)||f_n||_2^2 \leq \epsilon C(\beta)K a.s.| \\

Then, we want to show that \(|\Lambda_0 = \lim_{n \to 1} \langle f_n, \mathcal{H}_\beta f_n \rangle = \langle \tilde{f}, \mathcal{H}_\beta \tilde{f} \rangle \) a.s.

The second term in (1),

\[
\frac{2}{\sqrt{\beta}} \int_0^x f_n^2 B' dx \to \frac{2}{\sqrt{\beta}} \int_0^x \tilde{f}^2 B' dx \quad a.s.
\]

because \(f_n \to \tilde{f}\) uniformly on compacts.

We claim that the third term in (1),

\[
\frac{4}{\sqrt{\beta}} \int_0^x f'_n (\tilde{B} - B) dx \to \frac{4}{\sqrt{\beta}} \int_0^x \tilde{f} (\tilde{B} - B) dx \quad a.s.
\]

It’s enough to show \(\int_0^x |f_n f'_n - \tilde{f} \tilde{f}'| dx \to 0 a.s.
\)

Here we use triangle inequality. For large enough \(n\), we have

\[
\int_0^x |f_n f'_n - \tilde{f} \tilde{f}'| dx \\
\leq \int_0^x |f_n f'_n - \tilde{f} f'_n| dx + \int_0^x |\tilde{f} \tilde{f}' - \tilde{f} \tilde{f}'| dx \\
\leq \int_0^x |f_n|| f'_n - \tilde{f}'|dx + ||\tilde{f}||_2 ||\tilde{f} - f_n||_2 \\
\leq \int_0^x (|\tilde{f}| + \epsilon)| f'_n - \tilde{f}'|dx + ||f_n||_2 ||\tilde{f} - f_n||_2 \\
\to 0 \quad a.s.
\]

The last inequality is because of the uniformly convergence of \(\{f_n\}\) in \(L^2\).

Since \((|\tilde{f}| + \epsilon)\) is in \(L^2\) and \(f'_n \to \tilde{f}'\) weak convergence in \(L^2\), the first term in (2) \(\to 0 a.s.
\)

Since \(||f'||_2\) is finite and \(f_n \to f\) in \(L^2\), the second term in (2) \(\to 0 a.s.
\)

Then the claim follows.

Finally, we deal with the first term in (1).

By Fatou’s lemma,

\[
\liminf \int_0^\infty x f_n^2 dx \geq \int_0^\infty x \tilde{f}^2 dx \quad a.s.
\]

And

\[
||\tilde{f}'||_2^2 = \int_0^\infty \tilde{f} \tilde{f}' dx = \int_0^\infty f_n' \tilde{f}' dx \leq \liminf ||f'_n||_2 ||\tilde{f}'||_2 \quad a.s.,
\]

which implies

\[
\liminf \int_0^\infty (f'_n)^2 dx = \liminf ||f'_n||_2^2 \geq ||\tilde{f}'||_2^2 = \int_0^\infty (\tilde{f}')^2 dx \quad a.s.
\]
Hence,
\[
\tilde{\Lambda}_0 = \lim_{n \to \infty} \langle f_n, H_\beta f_n \rangle \\
= \lim_{n \to \infty} \left\{ \int_0^\infty \left[ (f_n')^2 + x f_n^2 \right] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f_n^2 B'dx + \frac{4}{\sqrt{\beta}} \int_0^\infty f_n f_n (B - B) dx + E \right\} \\
\geq \int_0^\infty \left[ (\tilde{f}')^2 + x \tilde{f}^2 \right] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty \tilde{f}^2 B'dx + \frac{4}{\sqrt{\beta}} \int_0^\infty \tilde{f} \tilde{f} (B - B) dx - c(\beta)K \\
\geq \langle \tilde{f}, H_\beta \tilde{f} \rangle - 2\epsilon C(\beta)K \text{ a.s.}
\]

Since \( \epsilon \) is arbitrary,
\[
\tilde{\Lambda}_0 \geq \langle \tilde{f}, H_\beta \tilde{f} \rangle \text{ a.s.}
\]

According to the definition of \( \tilde{\Lambda}_0 \),
\[
\tilde{\Lambda}_0 = \langle \tilde{f}, H_\beta \tilde{f} \rangle \text{ a.s.}
\]

Next, we prove the second part of the statement.
First, let’s go back to the matrices example. Assume that
\[
\lambda_0 = \inf_{||x||=1} \langle x, Ax \rangle = \min_{||x||=1} = \langle x_0, Ax_0 \rangle
\]
Then, for any \( \epsilon > 0 \)
\[
\frac{1}{||x_0 + \epsilon y||} \left( \langle x_0 + \epsilon y, A(x_0 + \epsilon y) \rangle - \langle x_0, A x_0 \rangle \right) \\
= (1 - 2\epsilon \langle x_0, y \rangle + O(\epsilon^2)) \langle x_0, A x_0 \rangle + 2\epsilon \langle y, A x_0 \rangle + O(\epsilon^2) - \langle x_0, A x_0 \rangle \\
= 2\epsilon \langle y, A x_0 \rangle - \langle x_0, y \rangle \langle x_0, A x_0 \rangle + O(\epsilon^2) \\
= 2\epsilon \langle y, A x_0 - \lambda_0 x_0 \rangle + O(\epsilon^2)
\]
Since \( x_0 \) is the minimizer, the first term in the above equation must be 0 for all \( y \). That is, \( \langle y, A x_0 - \lambda_0 x_0 \rangle = 0 \) for all \( y \). Hence, \( (\lambda_0, x_0) \) is an eigenvalue-eigenvector pair.
Then, we use the same idea for \( \tilde{f} \) and \( H_\beta \). Define that
\[
f^{\epsilon, \varphi} = \frac{\tilde{f} + \epsilon \varphi}{||\tilde{f} + \epsilon \varphi||_2}
\]
Then,
\[
\langle f^{\epsilon, \varphi} - H_\beta \tilde{f}^{\epsilon, \varphi}, \tilde{f}^{\epsilon, \varphi} \rangle - \langle \tilde{f}, H_\beta \tilde{f} \rangle \\
= 2\epsilon \left[ -\langle \tilde{f}, H_\beta \tilde{f} \rangle \int_0^\infty \tilde{f} \varphi dx + \int_0^\infty \langle \tilde{f}', \tilde{f} \varphi \rangle dx - \frac{2}{\sqrt{\beta}} \int_0^\infty \tilde{f} \varphi dx \right] \\
- \frac{2}{\sqrt{\beta}} \int_0^\infty (B - B) \langle \tilde{f} \varphi \rangle dx + O(\epsilon^2) \\
= 2\epsilon \langle \varphi, H_\beta \tilde{f} - \tilde{\Lambda}_0 \tilde{f} \rangle + O(\epsilon^2)
\]
Use the same argument that we used in matrices example, \( (\tilde{\Lambda}_0, \tilde{f}) \) is an eigenvalue-eigenfunction pair for \( H_\beta \). And we also proved the \( \tilde{\Lambda}_0 \) is the smallest eigenvalue.

\( \square \)
Define $\mathcal{H}_0$ as the function space of all the eigenfunction corresponding to $\widetilde{\Lambda}_0$.

**Claim** $\mathcal{H}_0$ is finite dimensional.

**Proof.** Suppose $\exists$ countable infinite $f_1, f_2, \cdots$, which form an orthonormal($L_2$) basis in $\mathcal{H}_0$. Then we can find a random variable $K$ so that $||f_i||^2(\text{a.s.}, \forall i \in \mathbb{N})$. By the lemma in the previous lecture, $\exists$ a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that $f_{n_k} \xrightarrow{L^2} f$ and $f \in \mathcal{H}_0$. Then $\forall n$, we can find $k$ such that $n_k \geq n$. Since $f_n \perp \{f_{n_k}, f_{n(k+1)}, \cdots\}$, we get $f_{n_k} \perp f$, which is a contradiction to the fact that $\{f_1, f_2, \cdots\}$ is an orthonormal basis. Hence, $\mathcal{H}_0$ is finite dimensional. \hfill \Box

We can define (find) the other eigenvalues inductively: $\widetilde{\Lambda}_0 \leq \widetilde{\Lambda}_1 \leq \widetilde{\Lambda}_2 \leq \cdots$.

### 1.2 Counting function of eigenvalues

First let’s look at a much simpler problem:

\[
\begin{align*}
-\partial_{XX} f &= \lambda f \\
f(0) &= 0, f(L) = 0
\end{align*}
\]

We have

\[f_{k-1}(x) = \sin(k \frac{\pi}{L} x), \quad k = 1, 2, \cdots\]

\[\lambda_{k-1} = (k \frac{\pi}{L})^2\]

Then, $k^{th}$ eigenfunction has $k$ roots in $(0, L)$.

For any $\lambda$:

\[
\begin{align*}
f_\lambda &\, : \, -\partial_{XX} f = \lambda f \\
f(0) &= 0
\end{align*}
\]

$\lambda$ is an eigenvalue $\iff f_\lambda(L) = 0$.

Then, $f_\lambda$ has $k$ roots in $(0, L) \iff \lambda_k \geq \lambda \geq \lambda_{k-1}$.

This is true in general on $[0, L]$:

\[
\begin{align*}
-\partial_{XX} f + V f &= \lambda f \\
f(0) &= 0
\end{align*}
\]

And for such kind problem, we can use the Riccait transformation:

\[p = \frac{f'}{f} \Rightarrow p' = \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} = \frac{p^2}{f} + \frac{f''}{f}\]

Then,

\[
\begin{align*}
-\partial_{XX} f + V f &= \lambda f \\
f(0) &= 0
\end{align*} \iff \begin{align*}
p' &= -\lambda + V - p^2 \\
p(0) &= \infty
\end{align*}
\]
# of explosions = # of eigenvalues below $\lambda$.
This is same on $[0, \infty)$:
\[
\begin{aligned}
\mathcal{H}_\beta f &= \lambda f \\
f(0) &= 0 
\end{aligned} \iff \begin{aligned}
-\partial_X X f + V f &= \lambda f \\
f(0) &= 0 
\end{aligned}
\]
where $V = x + \frac{2}{\sqrt{\beta}} B'(x)$.
Let $p = f'/f$, then
\[
p'(x) = \frac{2}{\sqrt{\beta}} B'(x) + x - \lambda - p^2(x)
\]
\[
\iff \, dp = \frac{2}{\sqrt{\beta}} dB + (x - \lambda - p^2) dx
\]
\[
\iff \, p(x_1) - p(x_0) = \frac{2}{\sqrt{\beta}} (B(x_1) - B(x_0)) + \int_{x_0}^{x_1} [x - \lambda - p^2(x)] dx
\]
The last equation is a diffusion model which has a general form
\[
dX_t = \sigma(X_t, t) dB_t + f(X_t, t) dt.
\]
The discrete analogue of the diffusion model is
\[
X_{t+\epsilon} = X_t + \eta_t
\]
where $\eta_t$ is asymptotically normal with expectation $f(X_t, t) \cdot \epsilon$ and variance $\sigma(X_t, t)^2 \cdot \epsilon$.
If $\sigma$ and $f$ are appropriate, it’s solvable. In our case, for the equation
\[
\mathcal{H}_\beta f = \lambda f, \quad f(0) = 0,
\]
if the solution $f$ decays fast enough, it can be proved that $(\lambda, f)$ is an ev-ef pair.

1.3 Limit of the matrix form

Recall that
\[
M_n^\beta = \begin{bmatrix}
a_n & b_{n-1} \\
b_{n-1} & a_{n-1} & b_{n-2} \\
& b_{n-2} & \ddots & \ddots \\
& & \ddots & \ddots & b_1 \\
& & & b_1 & a_1 
\end{bmatrix}, \quad H_n^\beta = n^{\frac{1}{2}} (2\sqrt{n} I_n - M_n)
\]
where $a_k \sim \frac{1}{\sqrt{\beta}} N(0, 2)$ and $b_k \sim \frac{1}{\sqrt{\beta}} \chi_{\beta k}$, and

$$H_\beta^\beta = n^{\frac{\beta}{2}}\begin{bmatrix} 2 & -1 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & \cdots & \cdots & \cdots & 2 \end{bmatrix} + n^{\frac{\beta}{2}}\begin{bmatrix} 0 & \gamma_1 & \cdots & \cdots & \cdots \\ \gamma_1 & 0 & \gamma_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{n-1} & \cdots & \cdots & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 2\rho_0 & \zeta_1 & \cdots & \cdots & \cdots \\ \zeta_1 & 2\rho_1 & \zeta_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \zeta_{n-1} & \cdots & \cdots & \cdots & 2\rho_{n-1} \end{bmatrix}$$

where $\gamma_k = \sqrt{n} - E\chi_{\beta(n-k)} \approx \frac{k}{2\sqrt{n}}$.

We know that

$$w_{n,1}(x) = n^{-\frac{1}{\beta}} \sum_{i=1}^{|x\cdot n^{1/3}|} \rho_i \Rightarrow B_1(x)$$

$$w_{n,2}(x) = n^{-\frac{1}{\beta}} \sum_{i=1}^{|x\cdot n^{1/3}|} \zeta_i \Rightarrow B_2(x)$$

$$2n^{-\frac{1}{\beta}} \sum_{i=1}^{|x\cdot n^{1/3}|} \gamma_i \rightarrow \frac{x^2}{2}$$

Define $||v||^2_{2,n} = n^{-\frac{1}{3}} \sum v_k^2$ and

$$\langle v, H_\beta^\beta v \rangle_{2,n} = n^{\frac{3}{2}} \sum (v_{k+1} - v_k)^2 + 2n^{-\frac{1}{\beta}} \sum \gamma_k v_k v_{k+1} + n^{-\frac{1}{\beta}} \sqrt{\frac{2}{\beta}} \sum \rho_k v_k^2$$

$$+ n^{-\frac{1}{\beta}} \sqrt{\frac{2}{\beta}} \sum \zeta_k v_k v_{k+1}$$

which are scaled version of $L_2$ norm and inner product of vectors, and

$$||v||^2 = n^{\frac{3}{2}} \sum (v_{k+1} - v_k)^2 + n^{-\frac{1}{3}} \sum k v_k^2 + n^{-\frac{1}{3}} \sum v_k^2$$

$$- n^{-\frac{1}{3}} \left( \frac{v_{k+1} - v_k}{n^{-1/3}} \right)^2 + n^{-\frac{1}{3}} \sum (k \cdot n^{-\frac{1}{3}}) v_k^2 + n^{-\frac{1}{3}} \sum v_k^2$$

which is a discrete version of $L^*$ norm if we view $v_k$ as the mesh sampled by size of $n^{-\frac{1}{3}}$.

**Lemma 1.** There exists a sequence of random constants $C_1(n), C_2(n), C_3(n)$ such that

$$C_1 ||v||^2_2 - C_2 ||v||^2_2 \leq \langle v, H_\beta^\beta v \rangle \leq C_3 ||v||^2_2.$$ 

Also $C_1(n), C_2(n), C_3(n)$ are tight.
Define \( L_{2,n} = \{ f \in L_2 : \text{step functions with mesh } n^{-\frac{1}{3}}, f(0) = 0, f(x) = 0, x \geq n^{\frac{2}{3}} \} \). For each \( f \in L_{2,n} \), it can be treated as a vector of length \( n \). Let \( P_n : L_2 \rightarrow L_{2,n} \) be the projection from \( L_2 \) to \( L_{2,n} \), then we can define

\[
\langle f, \hat{H}_n^\beta f \rangle_{L_2} = \langle P_n f, H_n^\beta P_n f \rangle_{2,n}.
\]

For each \( n \), let \( \lambda_n \) denote the minimum eigenvalue of \( \hat{H}_n^\beta \) and \( f_n \) denote the corresponding eigen function satisfying \( ||f_n|| = 1 \). We have that \( \lambda_n \rightarrow \lambda \) (maybe subsequence) and \( f_n \xrightarrow{L_2} f \).

Lemma 2.

1. If \( f_n \in L_{2,n} \) with \( f_n \rightarrow f \) in \( L_2 \) weakly, \( n^{\frac{1}{2}} (f_n(x + n^{-\frac{1}{2}}) - f_n(x)) \rightarrow f' \) in \( L_2 \) weakly, then for each \( \varphi \in C_0^\infty \),

\[
\langle \varphi, \hat{H}_n^\beta f_n \rangle \rightarrow \langle \varphi, H_\beta f \rangle
\]

Notice that the left term is a partial summation.

2. If \( f_n \in L_{2,n} \) is bounded in \( L_{*,n} \), then there exists a subsequence \( f_{n_k} \rightarrow f \) in \( L_2 \) such that

\[
\langle \varphi, \hat{H}_n^\beta f_n \rangle \rightarrow \langle \varphi, H_\beta f \rangle
\]

Therefore, for all \( \varphi \in C_0^\infty \),

\[
\lambda \langle \varphi, f \rangle = \lim \langle \varphi, \lambda_n f_n \rangle = \lim \langle \varphi, \hat{H}_n^\beta f_n \rangle = \langle \varphi, H_\beta f \rangle
\]

which implies \( (\lambda, f) \) is an ev-ef pair of \( H_\beta \).

Next, we want to show that \( \lambda \) is the smallest eigenvalue of \( H_\beta \). Let \( \Lambda_0 \) be the smallest eigenvalue, so there exists minimizing function \( f_0 \) for \( \Lambda_0 \). By definition the discretized version

\[
\lambda_n \leq \frac{\lim \langle f_{0,n}, \hat{H}_n^\beta f_{0,n} \rangle}{\lim \langle f_{0,n}, f_{0,n} \rangle} \rightarrow \Lambda_0
\]

where the left hand side \( \lambda_n \rightarrow \lambda \). So it implies that \( \lambda \leq \Lambda_0 \), and together with the fact that \( \lambda \geq \Lambda_0 \),

\[
\Rightarrow \lambda = \Lambda_0.
\]

We can use the induction to deal with other eigenvalues, then

\[
(\lambda_0^{(n)}, \lambda_1^{(n)}, \lambda_2^{(n)}, \ldots) \Rightarrow (\Lambda_0, \Lambda_1, \Lambda_2, \ldots)
\]

Remark: we can also obtain the edge scaling of the Laguerre ensemble \( M = AA^T = \frac{1}{\beta} BB^T \), where \( B \) is a bidiagonal matrix in the form

\[
B = \begin{bmatrix}
\chi_\beta m \\
\chi_\beta (m-1) \\
\chi_\beta (m-2) \\
\vdots \\
\end{bmatrix}
\]
2 Description of the limiting process

Recall that the density of eigenvalues are given by

\[ P_{\beta,n}(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{i>j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{4} \sum \lambda_i^2}, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_n. \]

2.1 Edge scaling limit

By Tracy-Widom, in the cases \( \beta = 1, 2, 4 \), \( n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \rightarrow TW_\beta = -\Lambda_0 \).

For any \( a > 0 \),

\[ P(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \geq a) = \exp(-\frac{2}{3} \beta a^3 (1 + o(1))) \tag{3} \]

\[ P(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \leq -a) = \exp(-\frac{1}{24} \beta a^3 (1 + o(1))). \tag{4} \]

For \( \beta = 1, 2, 4 \), one can prove this using the explicit formulas for the limits. By Ramirez-Rode-Virag, these asymptotic will hold for all \( \beta > 0 \).

\[ P(TW_\beta > a) = P(\Lambda_0 < -a) \]

where

\[ \Lambda_0 = \inf_{g \in L^*} \left\{ \int_0^\infty \left( f^2 + xf^2 \right) dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2 dB \right\} \]

\[ \leq \frac{1}{\|g\|_2^2} \left\{ \int_0^\infty (g^2 + xg^2) dx + \frac{2}{\sqrt{\beta}} \int_0^\infty g^2 dB \right\} \quad \forall g \in L^* \]

in which \( \frac{2}{\sqrt{\beta}} \int_0^\infty g^2 dB = N(0, 1) \times \frac{2}{\sqrt{\beta}} \left( \int_0^\infty g^4 dx \right)^{\frac{1}{2}} \). Therefore,

\[ P(\Lambda_0 < -a) \geq P \left( \frac{2}{\sqrt{\beta}} \left( \int_0^\infty g^4 dx \right)^{\frac{1}{2}} \times N(0, 1) \leq -a \int g^2 dx - \int (g^2 + xg^2) dx \right) \]

\[ = \Phi \left( -a \int g^2 dx - \int (g^2 + xg^2) dx \right) - \frac{2}{\sqrt{\beta}} (\int_0^\infty g^4 dx)^{\frac{1}{2}} \]

Since it holds for any \( g \in L^* \), we can choose a particular function \( g = \text{sech}(\sqrt{\beta}(x - 1)) \) to obtain the lower bound for up tail in (3). Similarly

\[ P(TW_\beta < -a) \leq P(\langle g, \mathcal{H}_\beta g \rangle > a \|g\|_2^2) \]

With a suitable test function, it gives you the upper bound in (4). And the other two inequality can be obtained by diffusion description.
2.2 Bulk scaling limit of the $\beta$-ensemble

We have showed that, for $|c| < 2$,

$$2\pi \sqrt{n} \sigma(c)(\Lambda_n - 2\sqrt{n}) \Rightarrow \text{some point process limit}, \quad \sigma(c) = \frac{1}{2\pi} \sqrt{4 - c^2}$$

By V-Viray, for each $\beta > 0$, there is a limiting point process(Sine$_\beta$) which does not depend on $c$.

How can we describe the process Sine$_\beta$?

We will describe the counting function $N(\lambda)$ of Sine$_\beta$.

$$N(\lambda) = \begin{cases} 
0 & \lambda = 0 \\
\#\{\text{points in } [0, \lambda]\} & \lambda > 0 \\
\#\{\text{points in } [-\lambda, 0)\} & \lambda < 0
\end{cases}$$

For any $a < b$, # of points in $[a, b) = N(b) - N(a) = N(b-a)$ which means it is translation invariant.

Next we will introduce two equivalent description of the process.

2.2.1 First description

Let $Z(t)$ be complex BM. $Z(t) = B_1(t) + iB_2(t)$ where $B_1, B_2$ are independent standard BM.

Consider the following family of stochastic differential equations.

$$d\alpha_\lambda(t) = \lambda \beta^4 e^{-\frac{\beta}{4}t} dt + \Re \left((e^{-i\alpha_\lambda} - 1) dZ\right), \quad \alpha_\lambda(0) = 0$$

where $\Re \left((e^{-i\alpha_\lambda} - 1) dZ\right) = (\cos \alpha_\lambda - 1) dB_1 + \sin \alpha_\lambda dB_2$.

$$\alpha_\lambda(t) = \int_0^t \lambda \beta^4 e^{-\frac{\beta}{4}s} ds + \int_0^t \Re \left((e^{-i\alpha_\lambda(s)} - 1) dZ\right)$$

Claim:

1. This will have a solution.

2. $\frac{1}{2\pi} \lim_{t \to \infty} \alpha_\lambda(t)$ exists a.s. and it’s an integer. Define

$$N(\lambda) = \frac{1}{2\pi} \lim_{t \to \infty} \alpha_\lambda(t).$$

3. $N(\lambda)$ is increasing in $\lambda$. $N(\lambda)$ is an integer-valued step function.

4. $N(\lambda)$ is the counting function of Sine$_\beta$. 

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2.2.2 Second description

In this section, we need some knowledge about 2D hyperbolic space. Specifically, we will use the Poincaré disk model on $U = \{ |z| < 1 \}$.

- **lines:**
  1) lines passing 0
  2) circles $\perp \{ |z| = 1 \}$

- **angles:** same as in the euclidean plane

- **distance:** $4 \frac{1}{1-|z|^2}$

It’s conformal invariant. For $|\alpha| < 1$, the transformation $T(z) = e^{iv} \frac{z - \alpha}{1 - \overline{\alpha}z}$ keep distance, angle, \ldots

**Rotation around a point.**

- rotation around 0: $z \rightarrow e^{iv} \cdot z$
- rotation around $\alpha(|\alpha| < 1)$: first shift $T(z) = \frac{z - \alpha}{1 - \overline{\alpha}z}$, then rotation and transform back
  $$z \rightarrow T^{-1}(e^{iv} \cdot T(z))$$

**BM in the hyperbolic plane - scaling limit of a simple random walk in the hyperbolic plane.**

$$dB = \frac{1 - |B|^2}{2} dZ$$
where $B(t)$ is hyperbolic BM and $Z(t)$ is complex BM.

For each $\lambda$, we will follow a point $Z_\lambda(t) \in U$ on the boundary $z_\lambda(0) = 1$. We rotate (in hyp) $Z_\lambda(t)$ about the point $B(t)$ with speed $\lambda \frac{\beta}{2} e^{-\frac{\beta}{2}t}$, and

$$\frac{\partial Z_\lambda(t)}{\partial t} = i \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \frac{z_\lambda - B}{1 - |B|^2}$$

For each $\lambda$, we also follow the winding angle of $z_\lambda$ about $B(t)$.

$$\alpha_\lambda(t) = \langle 1, B(t), Z(t) \rangle$$

Define $N(\lambda)$ as the total winding number of $Z_\lambda(t)$.

**Claim:**

1. $N(\lambda)$ is increasing.
2. $N(\lambda)$ is the counting function of Sine$_\beta$. 