The First Mayr-Meyer Ideal

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Introduction

In a 1926 paper Grete Hermann proved that for any ideal I in an n-dimensional polynomial ring over the field of rational numbers, if I is generated by polynomials $f_1, \ldots, f_k$ of degree at most d, then it is possible to write $f = \sum r_i f_i$ where each r_i has degree at most deg $f + (kd)^{(2^n)}$. Almost fifty years later Ernst Mayr and Albert Meyer found a class of ideals $J(n, d)$ for which a doubly exponential bound in n is indeed achieved. One of the questions pertaining to these Mayr-Meyer ideals is the underlying cause for the doubly exponential behavior: is the behavior due to the number of minimal primes, to the number of associated primes, or to the structure of one of them? One of the approaches in tackling this problem is to give an explicit primary decomposition of these ideals. For $n > 1$, this approach proves to be computationally quite hard even for a computer algebra system.

Prof. Irena Swanson has given an explicit decomposition and answered the question completely for the case $n = 1$ when the underlying field is extended to contain the d^{th} roots of unity. We give a similar decomposition when the underlying field is the rationals and no extra roots of unity are thrown in.

The first Mayr-Meyer ideal

The first Mayr-Meyer ideal $J(1, d)$ is defined as follows. Let d be a positive integer, let $s, f, s_1, f_1, c_1, \ldots, c_4, b_1, \ldots, b_4$ be indeterminates over the rationals, and $R = \mathbb{Q}[s, f, s_1, f_1, c_1, \ldots, c_4, b_1, \ldots, b_4]$. Note that R has dimension 12. The Mayr-Meyer ideal for $n = 1$ is the ideal in R with generators as follows:

$$\begin{aligned} J = J(1,d) = \ & (s_1 - sc_1, f_1 - sc_4) + (c_i(s - fb_i^d) | i = 1,2,3,4) \\ & + (fc_1 - sc_2, fc_4 - sc_3, s(c_3 - c_2), f(c_2b_1 - c_3b_4), fc_2(b_2 - b_3)) \end{aligned}$$

Experimental results and patterns

Some experimental results and patterns obtained using the Macaulay 2 computer algebra system.

d	components
3	7
4	8
5	7
6	9
7	7
8	9
9	8
10	9
11	7
12	11
13	7
14	9
15	9
16	10

Typical decompositions when d is a prime

$$\begin{aligned} J(1,5) = \ & \langle s_1 - sc_1, f_1 - sc_4, c_1, c_2, c_3, c_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^5, b_1 - b_4, b_2 - b_3, b_1 - b_2 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^5, b_1 - b_4, b_2 - b_3, \\ & \quad b_1^4 + b_1^3 b_2 + \ldots + b_2^4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, f \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_2, c_3, b_3^5, b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_4, b_3^5, b_2 - b_3, c_2b_1 - c_3b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s^2, f^2, c_4(s - fb_4^5), c_3(s - fb_3^5), sc_3 - fc_4, c_3^2, c_4^2, \\ & \quad c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4 \rangle \end{aligned}$$

$$\begin{aligned} J(1,7) = \ & \langle s_1 - sc_1, f_1 - sc_4, c_1, c_2, c_3, c_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^7, b_1 - b_4, b_2 - b_3, b_1 - b_2 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^7, b_1 - b_4, b_2 - b_3, \\ & \quad b_1^6 + b_1^5 b_2 + \ldots + b_2^6 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, f \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_2, c_3, b_3^7, b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_4, b_3^7, b_2 - b_3, c_2b_1 - c_3b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s^2, f^2, c_4(s - fb_4^7), c_3(s - fb_3^7), sc_3 - fc_4, c_3^2, c_4^2, \\ & \quad c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4 \rangle \end{aligned}$$

Our proposition

A minimal primary decomposition of $J(1, p)$ over $\mathbb{Q}$, for p prime, is:

$$\begin{aligned} J(1,p) = \ & \langle s_1 - sc_1, f_1 - sc_4, c_1, c_2, c_3, c_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^p, b_1 - b_4, b_2 - b_3, b_1 - b_2 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^p, b_1 - b_4, b_2 - b_3, \\ & \quad b_1^{p-1} + b_1^{p-2} b_2 + \ldots + b_2^{p-1} \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, f \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_2, c_3, b_3^p, b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s, c_1, c_4, b_3^p, b_2 - b_3, c_2b_1 - c_3b_4 \rangle \\ & \cap \langle s_1 - sc_1, f_1 - sc_4, s^2, f^2, c_4(s - fb_4^p), c_3(s - fb_3^p), sc_3 - fc_4, c_3^2, c_4^2, \\ & \quad c_1 - c_4, c_2 - c_3, b_2 - b_3, b_1 - b_4 \rangle \end{aligned}$$

Proof. Now all the above ideals except the third one we know are primary from [I]. To show that the third one is primary we need to show that the following polynomial is irreducible over the rationals:

$$b_1^{p-1} + b_1^{p-2} b_2 + \ldots + b_2^{p-1}$$

Note that it suffices to prove that the polynomial $q(x) = \frac{x^p - 1}{x - 1}$ is irreducible over the rationals. Now:

$$q(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + 1$$

Now if we can show that $q(x+1)$ is irreducible we are done.

$$q(x+1) = p + \sum_{i=1}^{p-1} \binom{i}{1} x + \sum_{i=2}^{p-1} \binom{i}{2} x^2 + \ldots + x^{p-1}$$

But note that $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ for $0 \le r \le n$. Hence:

$$q(x+1) = p + \binom{p}{2} x + \binom{p}{3} x^2 + \binom{p}{4} x^3 + \ldots + x^{p-1}$$

Now as p is prime we have that p divides each $\binom{p}{i}$ for $0 < i < p$. Hence applying Eisenstein's criterion we have that $q(x+1)$ is irreducible over the rationals.
Again building on [I] if we can show that the intersection of the second and third ideals in the decomposition is:

$$\langle s_1 - sc_1, f_1 - sc_4, c_4 - c_1, c_3 - c_2, c_1 - c_2b_1^p, b_1 - b_4, b_2 - b_3, b_1^p - b_2^p \rangle$$

then we are done, but this is quite evidently true as both the ideals are coprime and thus their intersection is just their product which gives the above ideal.

Open questions and conjectures

1. Experimental results suggest that over $\mathbb{Q}$ the total number of components in the primary decomposition of $J(1, d)$ is $5 + div(d)$, where $div(d)$ denotes the total number of divisors of d.
2. Does a similar pattern carry over for arbitrary $J(n, d)$
3. Our work so far shows that the doubly exponential behaviour for $n = 1$ is not due to the number of components but due to the structure of one or more of them, does the same hold true for higher values of n?
4. If the answer to the above question is yes (as it seems likely) then is it the structure of the same associated prime for all n that causes the doubly exponential behaviour?

References and acknowledgements

[I] Irena Swanson, The first Mayr-Meyer ideal, Preprint 2001.

A special thanks to Prof. Irena Swanson for introducing us to the Mayr-Meyer ideals and helping us understand them.