Stability of Shear Flow

Fabian Waleffe, written up by Zhan Wang and Sam Potter
Revised by FW
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Abstract
A look at energy stability, valid for all amplitudes, and linear stability for shear flows.

1 Nonlinear stability

Associated Navier-Stokes equation:
\[ \partial_t v + v \cdot \nabla v + \nabla P = F + \nu \nabla^2 v \quad \text{with} \quad \nabla \cdot v = 0 \] (1)

In this equation \( \nu = R^{-1} \) is the nondimensional viscosity coefficient, where \( R \) is the Reynolds number. Let us assume a base flow \( U(x, t) \) that is a known solution to equation (1) driven by the body force \( F \) (e.g., an imposed pressure gradient \( F = -\hat{x} dP_0/dx \) in channel flow, or gravity for flow down an inclined channel) and/or the boundary conditions. Next we perturb the flow as \( v = U + u \) where \( u = (u, v, w) \) represents the perturbation. We plug this \( v \) into equation (1) which yields:
\[ \partial_t (U + u) + U \cdot \nabla u + u \cdot \nabla U + u \cdot \nabla u + \nabla P + \nabla (P + p) = F + \nu \nabla^2 (U + u) \] (2)

Since \( U \) is a solution of equation (1) the associated terms cancel and we get the perturbation equation:
\[ \partial_t u + U \cdot \nabla u + u \cdot \nabla U + u \cdot \nabla u + \nabla p = \nu \nabla^2 u \] (3)

with the incompressible constraint \( \nabla \cdot u = 0 \). For the domain \( V \) with fixed boundary \( \partial V \), the boundary condition for \( u \) is homogeneous, namely, \( u|_{\partial V} = 0 \) or periodic. Note that the decomposition \( v = U + u \) into a base flow plus a perturbation is different from the Reynolds decomposition \( v = \bar{v} + v' \) into a mean plus a fluctuation. The base flow \( U \) is a solution of the Navier-Stokes equations and is independent of the perturbation \( u \), but the mean flow is \( \bar{v} \) is not a solution of Navier-Stokes and is coupled to the fluctuations \( v' \) through the Reynolds stresses.

In order to calculate the total kinetic energy of the perturbation, we multiply equation (3) by \( u \) and integrate over the domain \( V \)
\[ \int_V u \cdot \left( \partial_t u + U \cdot \nabla u + u \cdot \nabla U + u \cdot \nabla u + \nabla p - \nu \nabla^2 u \right) dV = 0 \] (4)
Direct computation using integration by parts and the incompressibility condition ($\nabla \cdot U = 0 \rightarrow \nabla \cdot u = 0$) yields

$$\frac{d}{dt} \int_V \frac{|u|^2}{2} dV = \int_V -u \cdot \nabla U \cdot u dV - \nu \int_V \nabla u : \nabla u^T dV$$

$$\triangleq \int_V -u \cdot S \cdot u dV - \nu \int_V |\nabla u|^2 dV$$  \hspace{1cm} (5)

where $S$ is the symmetric tensor strain rate tensor defined as $S_{ij} = \frac{1}{2} \left( \partial_i U_j + \partial_j U_i \right)$ and $u_i S_{ij} u_j = u_i (\partial_j U_j) u_j$ using Einstein summation and $\nabla u : \nabla u^T \triangleq (\partial_i u_j)(\partial_i u_j) = |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \triangleq |\nabla u|^2$. Since the dissipation term is always positive, if the production term is negative or zero the flow is absolutely stable, that is, stable to any perturbation $u$.

Example: Rigid body rotation is absolutely stable, since the production term is 0. In this case

$$U = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \nabla U = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{ij} = \frac{1}{2} (\partial_i U_j + \partial_j U_i) \Rightarrow S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (6)

Definition (growth rate): We can think of the right hand side of (5) normalized by $2E \triangleq \int_V |u|^2 dV$ as a growth rate since if the perturbation had the form $u \triangleq e^{\lambda t} \tilde{u}(x)$, as would be the case for time independent $U$ in the linear limit, we would have $|u|^2 = \exp(2\sigma t)|\tilde{u}|^2$ and $(2E)^{-1} dE/dt = \sigma = \Re(\lambda)$, so $\sigma$, the real part of $\lambda$, is called the growth rate. We have $-u \cdot S \cdot u \leq \lambda_{\text{max}} u \cdot u$ where $\lambda_{\text{max}}$ is the largest eigenvalue of the real and symmetric ($-S$). Manipulating the right hand side of equation (5) gives

$$\sigma \triangleq \frac{\int_V -u \cdot S \cdot u dV}{\int_V |u|^2 dV} - \nu \frac{\int_V |\nabla u|^2 dV}{\int_V |u|^2 dV}$$

$$\leq \frac{\lambda_{\text{max}} \int_V |u|^2 dV}{\int_V |u|^2 dV} - \nu \frac{\int_V |\nabla u|^2 dV}{\int_V |u|^2 dV} \leq \lambda_{\text{max}}$$  \hspace{1cm} (7)

and this provides a simple upper bound on the growth rate of any instability.

Theorem (Serrin 1959): For any steady solution $U$ there exists a critical Reynolds number $Re_1 > 0$ such that for any flow with $Re \leq Re_1$, the system is absolutely stable. See [2, §53.1] or [1, §9.6].

Next let’s turn to shear base flows, i.e. $U = U(y) \hat{x}$ and

$$S = \begin{pmatrix} 0 & \frac{U''}{2} & 0 \\ \frac{U'}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad U' \triangleq \frac{dU}{dy}.$$  \hspace{1cm} (8)

The corresponding kinetic energy takes the form

$$\frac{d}{dt} \int_V \frac{|u|^2}{2} dV = \int_V -uv \frac{dU}{dy} dV - \nu \int_V |\nabla u|^2 dV$$  \hspace{1cm} (9)
that is very similar to the fluctuation energy equation derived in lecture 2, but again
the production term there involved the mean shear rate $dU/dy$ that depends on the
Reynolds stress $\overline{uv}$, while here we have the base shear rate $dU/dy$ that is independent
of $uv$. For shear flows, the growth rate $\sigma < \max(U'/2)$ (assuming $U' \geq 0$) and this
maximum would require very large Reynolds numbers $\nu = 1/R \to 0$ and $u = -v$ with
$w = 0$, localized near the max of $U'$. In the case of nondimensional Couette flow
$U(y) = y$, the energy equation reads
\[
\frac{d}{dt} \int_V \frac{|u|^2}{2} dV = \int_V -uv dV - \nu \int_V |\nabla u|^2 dV \quad (10)
\]
From this equation it can be seen that $-uv > 0$ occurring somewhere in the domain $V$
is a necessary condition for instability. Turning to the energy stability of shear flows, if
we define the critical value $\nu_E$
\[
\nu_E \equiv \max \frac{\int_V -uv dV}{\nu \int_V |\nabla u|^2 dV} \quad (11)
\]
it directly follows that
\[
\frac{d}{dt} \int_V \frac{|u|^2}{2} dV \leq (\nu_E - \nu) \int_V |\nabla u|^2 dV. \quad (12)
\]
The inequality (12) shows that the perturbation is stable if $\nu_E < \nu \Leftrightarrow R < 1/\nu_E \equiv R_E$. This is a sufficient condition for stability and is known as the absolute stability threshold. Therefore an argument for absolute stability turns into an optimization problem (11)
with the constraints $\nabla \cdot u = 0$ and $u \big|_{\partial V} = 0$

Remark: For Couette flow, the critical Reynolds number for absolute stability is about
20.7, see [2, §53.1].

2 Linear stability
The flow is decomposed into a base flow $U$ and a perturbation about the base flow $u$
\[
v = U + u. \quad (13)
\]
Plugging into the Navier-Stokes equations and neglecting the quadratic nonlinearity
$u \cdot \nabla u$ gives
\[
\partial_t u + U \cdot \nabla u + u \cdot \nabla U + \nabla p = \nu \nabla^2 u. \quad (14)
\]
The base flow is now taken to be a shear flow, $U = U(y) \hat{x}$. Taking the curl of (14) and
dotting with the vertical unit vector $\hat{y}$ gives
\[
(\partial_t + U \partial_x - \frac{1}{R} \nabla^2) \eta = -\partial_z v \frac{dU}{dy} \quad (15)
\]
where $\eta = \hat{y} \cdot \nabla \times u$ is the vertical component of the vorticity and $v = \hat{y} \cdot u$ is the vertical
component of the perturbation velocity. Taking the curl of (14) twice and dotting with
the vertical unit vector gives
\[
(\partial_t + U \partial_x - \frac{1}{R} \nabla^2) \nabla^2 v - \partial_x v \frac{d^2 U}{dy^2} = 0. \quad (16)
\]
Equations (15) and (16) are known as the Squire and Orr-Sommerfeld equations, respectively. Note that the \( v \) equation (16) is decoupled from the \( \eta \) equation (15). There are two basic kinds of boundary conditions at the walls of the channel. One is no-slip boundary condition \( u = v = w = 0 \) which implies there is no perturbation at the walls. In the Orr-Sommerfeld equation this boundary condition takes the form \( v = 0 \), \( v_y = -u_x - w_z = 0 \) and \( \eta = u_x - w_y = 0 \). The other is ‘free slip’ boundary conditions (i.e. stress or Neumann boundary conditions on the full flow) \( v = u_y = w_y = 0 \) which implies \( v = 0 \), \( \eta_y = u_yz - w_xz = 0 \) and \( v_yk = -u_yz - w_yz = 0 \) at the walls of the channel. Next we turn to the Fourier analysis of the Orr-Sommerfeld system, since \( U = U(y) \) only, equations (15) and (16) admit solutions of the form

\[
\eta(x, y, z, t) = \hat{\eta}(y)e^{i\lambda z}e^{i(\alpha x + \gamma z)}
\]

\[
v(x, y, z, t) = \hat{v}(y)e^{i\lambda z}e^{i(\alpha x + \gamma z)}
\]

where \( \lambda \) is a complex-valued growth rate, \( \alpha \) and \( \gamma \) are the real streamwise and spanwise wavenumbers, respectively, and \( \hat{\eta}(y) \) and \( \hat{v}(y) \) are complex functions. Plugging the above forms of \( v \) and \( \eta \) into equations (15) and (16) gives

\[
\begin{align*}
\left[ \lambda + i\alpha U - \frac{1}{R}(D^2 - k^2) \right] \hat{\eta} &= -i\gamma \hat{v}U' \\
\left[ \lambda + i\alpha U - \frac{1}{R}(D^2 - k^2) \right] (D^2 - k^2) \hat{v} - U''i\alpha \hat{v} &= 0
\end{align*}
\]

(17)

(18)

where a prime indicates a \( y \)-derivative, \( D = d/dy \), and \( k^2 = \alpha^2 + \gamma^2 \). Equation (18) can be simplified by multiplying through by \( k/\alpha \)

\[
\left[ \tilde{\lambda} + ikU - \frac{1}{\tilde{R}}(D^2 - k^2) \right] (D^2 - k^2) \hat{v} - U''ik \hat{v} = 0
\]

(19)

where \( \tilde{\lambda} = \lambda k/\alpha \) and \( \tilde{R} = R\alpha/k \).

**Squire’s theorem.** Equation (19) is (18) with \( \alpha = k \) and rescaled growth rate and Reynolds number. Therefore a three dimensional perturbation with wavenumbers \((\alpha, \gamma)\) at Reynolds number \( R \) with growth rate \( \lambda \) is mathematically equivalent to a two dimensional perturbation with wavenumbers \((k, 0)\) but with Reynolds number \( \tilde{R} = R\alpha/k \leq R \) and growth rate \( \Re(\tilde{\lambda}) = \Re(\lambda)k/\alpha \geq \Re(\lambda) \). In other words, for any 3D unstable mode \((\alpha, \gamma)\) there is a 2D unstable mode \((\sqrt{\alpha^2 + \gamma^2}, 0)\) with larger growth rate at a lower Reynolds number. This is Squire’s Theorem [3], [2]. Another way to derive this result, is to let \( \alpha U(y) = k \tilde{U}(y) \) in (19) and conclude that a 3D perturbation is equivalent to a 2D perturbation with a weaker shear flow \( \tilde{U}(y) = \alpha U(y)/k \).

Furthermore, it is easy to show that the homogeneous \( \eta \) equation, that is (17) with \( v = 0 \) has only damped modes (multiply the homogeneous equation by \( \eta^* \) the conjugate of \( \eta \), integrate from wall to wall and add the complex conjugate of the result to show that \( \lambda + \lambda^* = 2\sigma \leq 0 \)). This is physically obvious since (17) is an advection diffusion equation when \( v = 0 \), in fact the decay of \( \int_V \eta^2 dV \) for \( v = 0 \) can be shown for the full linear PDE (15). Thus the eigenvalue problem for (15), (16), reduces to the consideration of the Orr-Sommerfeld equation (19) for 2D perturbations only. Note that equation (17) with \( v \neq 0 \) exhibits transient growth for 3D perturbations with \( \partial_z v \neq 0 \), as discussed in lecture 1. The physical mechanism behind this is the redistribution of streamwise velocity \( U(y) \) by the perturbation \( v \) to create \( u \) perturbations and \( \eta = \partial_z u - \partial_z w \).
3 Energy equation

From Squire’s theorem it suffices to consider (18) with \( k = \alpha \)
\[
\left[ \lambda + i\alpha U - \frac{1}{R}(D^2 - \alpha^2) \right] (D^2 - \alpha^2) \hat{\nu} - U''i\alpha\hat{\nu} = 0. \tag{20}
\]
The equation for the complex conjugate \( \hat{\nu}^* \) reads
\[
\left[ \lambda^* - i\alpha U - \frac{1}{R}(D^2 - \alpha^2) \right] (D^2 - \alpha^2) \hat{\nu}^* + U''i\alpha\hat{\nu}^* = 0. \tag{21}
\]
since \( U(y) \), \( \alpha \) and \( R \) are real and where \( \lambda^* \) is the complex conjugate of \( \lambda \). Doing the following surgery: \( \hat{\nu}^* \cdot (20) + \hat{\nu} \cdot (21) \) and integrating from the bottom of the domain \( (y = y_1) \) to the top of the domain \( (y = y_2) \) yields
\[
(\lambda + \lambda^*) \int_{y_1}^{y_2} (|D\nu|^2 + \alpha^2|\nu|^2) dy = \int_{y_1}^{y_2} U'Tdy - \frac{2}{R} \int_{y_1}^{y_2} |\phi|^2 dy. \tag{22}
\]
Here \( T \triangleq i\alpha(\hat{\nu}D\hat{\nu}^* - \hat{\nu}^*D\hat{\nu}) \equiv -\alpha^2\bar{\nu} \), and \( \phi \triangleq (D^2 - \alpha^2)\hat{\nu} \) with
\[
\int_{y_1}^{y_2} |\phi|^2 dy = \int_{y_1}^{y_2} |D^2\nu - \alpha^2\nu|^2 dy = \int_{y_1}^{y_2} (|D^2\nu|^2 + 2\alpha^2|D\nu|^2 + \alpha^4|\nu|^2) dy \geq 0. \tag{23}
\]
It is noted that when doing the integration by parts, the boundary condition \( \hat{\nu}\big|_{\partial\Omega} = 0 \) is used, and either \( D\hat{\nu}\big|_{\partial\Omega} = 0 \) or \( D^2\hat{\nu}\big|_{\partial\Omega} = 0 \) can be applied to lead the same equality (22). In other words, (22) holds for both no-slip and free-slip boundary conditions.

Equation (22) is simply the version of (9) for 2D eigensolutions \( \nu = \hat{\nu}(y)e^{\lambda t}e^{i\alpha x} \) with \( u = \hat{u}(y)e^{\lambda t}e^{i\alpha x} = (i/\alpha)D\hat{\nu} e^{\lambda t}e^{i\alpha x} \) to satisfy \( \partial_x u + \partial_y v = 0 \) and the ‘Reynolds stress’ for such a perturbation is
\[
-\bar{\nu}\nu = -(\hat{\nu}^* \hat{\nu} + \hat{\nu} \hat{\nu}^*) e^{2\sigma t} = \frac{i}{\alpha} (\hat{\nu}D\hat{\nu}^* - \hat{\nu}^*D\hat{\nu}) e^{2\sigma t} \equiv \frac{T}{\alpha^2} e^{2\sigma t} \tag{24}
\]
where \( 2\sigma = \lambda + \lambda^* \) and \( \bar{\nu}\nu \) is the horizontal average of uv. Likewise, \( \phi \triangleq (D^2 - \alpha^2)\hat{\nu} \) is effectively the \( z \) component of vorticity \( \omega_z = \partial_x v - \partial_y u = (i/\alpha)(i\alpha)(D^2\nu) e^{M t}e^{i\alpha x} = (-i/\alpha)(i\alpha)(\nu) e^{M t}e^{i\alpha x} \).

For free-slip boundary conditions, \( \hat{\nu} = D^2\hat{\nu} = 0 \) at the walls, we can derive a useful form of the enstrophy equation, i.e. an equation for the integral of vorticity squared. Consider \( \int_{y_1}^{y_2} [(D^2 - \alpha^2)\hat{\nu}^* \cdot (20) + (D^2 - \alpha^2)\hat{\nu} \cdot (21)] dy = 0 \) to obtain
\[
(\lambda + \lambda^*) \int_{y_1}^{y_2} |\phi|^2 dy = -\int_{y_1}^{y_2} U''Tdy - \frac{2}{R} \int_{y_1}^{y_2} [D\phi|^2 + \alpha^2|\phi|^2] dy \geq 0 \tag{25}
\]
It follows directly that \( U'' = 0 \) implies linear stability for free-slip, that is \( 2\sigma = \lambda + \lambda^* \leq 0 \). In other words plane Couette flow \( U(y) = y \) and plane Poiseuille flow \( U(y) = 1 - y^2 \), in \(-1 \leq y \leq 1 \), are linearly stable for free-slip (i.e. imposed stress) boundary conditions as well as any combination of Couette and Poiseuille \( U(y) = a + by + cy^2 \), among other flows. We’ll discuss this further in the next lecture.

The enstrophy equation (25) only holds under free-slip boundary condition, since \( \hat{\nu}\big|_{\partial\Omega} = D^2\hat{\nu}\big|_{\partial\Omega} = 0 \) leads to the cancelation of the boundary terms in integration by parts but a boundary term of indefinite sign subsists for no-slip \( \hat{\nu} = D\hat{\nu} = 0 \) at the walls. The physical meaning of these boundary terms is that vorticity can be generated (or destroyed) at the walls for no-slip but not for free-slip.
References

