Stability of Shear Flow: part 2

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Abstract

We derive necessary conditions for linear instability of shear flows and prove linear stability of plane Couette, Poiseuille and Kolmogorov flows for *viscous* flow with *stress* boundary conditions (*i.e.* free-slip perturbations) thereby generalizing well-known *inviscid* stability results. We give a straightforward derivation of classic inviscid results by combining the perturbation energy and enstrophy equations. We then summarize the stability of various canonical shear flows and conclude the implications of energy stability and linear theory. Furthermore, we examine inflectional instabilities and introduce their role in the self-sustaining process.

1 Necessary conditions for linear instability

In the previous lecture the full flow $\mathbf{v}$ has been decomposed into a base shear flow $U(y)\hat{\mathbf{x}}$ and a perturbation $\mathbf{u}$. The Navier-Stokes equations have been linearized about the base flow $U(y)\hat{\mathbf{x}}$ and this led us to the Squire and Orr-Sommerfeld equation after elimination of the pressure. Since the equations for $\mathbf{u}$ have been linearized and its coefficients depend only on $U(y)$, we can reduce the solution to the consideration of perturbations of the form $\mathbf{u} = \hat{\mathbf{u}}(y)e^{\lambda t}e^{i\alpha x}e^{i\gamma z}$ with α, γ real (*i.e.* Fourier-Laplace expansion of $\mathbf{u}$). Then *Squire's theorem* shows that it suffices to consider 2D perturbations ($\gamma = 0$) to investigate exponentially growing modes, that is solutions with $2\Re(\lambda) = \lambda + \lambda^* = 2\sigma > 0$. We define $\lambda = \sigma - i\omega$ where $i^2 = -1$ and σ and ω are real.

We derived an energy and enstrophy equation for those linear 2D perturbations and both equations include a production term that involves the perturbation 'Reynolds stress' $-\overline{uv} \equiv \alpha^{-2}T(y)e^{2\sigma t}$, where

$$T(y) = -\alpha^2\left(\hat{u}^*\hat{v} + \hat{u}\hat{v}^*\right) = i\alpha\left(\hat{v}D\hat{v}^* - \hat{v}^*D\hat{v}\right), \tag{1}$$

such that $T = 0$ at the walls at $y = y_1$ and $y = y_2$ since $v = 0$ there. We write $D \equiv d/dy$ for compactness. We drop the hat over $\hat{v}$ below.

The perturbation energy equation derived in the previous lecture implies that for an instability, $\sigma > 0$, we must have

$$2\sigma\int\left(|Dv|^2 + \alpha^2|v|^2\right) + \frac{2}{R}\int|\phi|^2 = \int U'T = -\int\left(U - U_0\right)T' > 0, \tag{2}$$

where $\int(\cdots)$ is short for $\int_{y_1}^{y_2}(\cdots)dy$, the integral from the bottom wall at $y = y_1$ to the top wall at y_2, the prime $(\cdot)' \equiv D(\cdot) \equiv d(\cdot)/dy$, that is $U' = dU/dy$, $U'' =$

d^2U/dy^2,... The function $\phi = (D^2 - \alpha^2)v$ is effectively the perturbation vorticity (see lecture 3).[1]

Equation (2) follows from multiplying the Orr-Sommerfeld equation (17) by v^*, integrating over the full channel from $y = y_1$ to y_2 using integration by parts and taking the real part of the result. The last expression in (2) was obtained by integration by parts of $\int U'T$ and U_0 is an arbitrary constant since $\int U_0 T' = U_0 \int T' = 0$ because $T = 0$ at the walls.

For *free-slip* boundary conditions, that is $v = D^2v = 0$ at the walls (corresponding to stress boundary conditions on the full flow, that is $\mathbf{v} \cdot \hat{\mathbf{y}} = 0$ with $\partial_y \mathbf{v}_\parallel$ fixed), the perturbation enstrophy equation derived in lecture 3 reads

$$2\sigma \int |\phi|^2 + \frac{2}{R} \int \left(|D\phi|^2 + \alpha^2 |\phi|^2\right) = \int U'' T' = \int (-U''')T \ > 0, \tag{3}$$

and the enstrophy production $\int U''T' = \int (-U''')T$ should be positive for an instability $\sigma > 0$. This equation was obtained by multiplying the Orr-Sommerfeld equation (17) by $\phi^* = (D^2 - \alpha^2)v^*$, integrating over the channel using multiple integrations by parts then taking the real part of the integral equation (*i.e.* adding its complex conjugate). This yields the enstrophy equation (3) together with the boundary term

$$\frac{1}{R}\Big[\phi^* D\phi + \phi D\phi^*\Big]_{y_1}^{y_2} \tag{4}$$

on the right hand side of (3). This boundary term vanishes for $v = D^2v = 0$ on the boundary since $\phi = D^2v - \alpha^2 v$. For no-slip, $v = Dv = 0$, the boundary term (4) is sign indefinite and corresponds to the generation or destruction of enstrophy at the walls.

Now, since $\phi = (D^2 - \alpha^2)v$, integration by parts with $v = 0$ at the walls gives

$$\int |\phi|^2 = \int \left(|D^2v|^2 + 2\alpha^2 |Dv|^2 + \alpha^4 |v|^2\right) \tag{5}$$

so we can combine the energy and enstrophy equation taking (3) - α^2 (2) to obtain

$$\begin{aligned} 2\sigma \int \left\{|D^2v|^2 + \alpha^2 |Dv|^2\right\} + \frac{2}{R}\int |D\phi|^2 &= \\ -\int (U''' + \alpha^2 U')T &= \int \left(U'' + \alpha^2(U - U_0)\right) T' \ > 0. \end{aligned} \tag{6}$$

We can go even further and take (6) - β^2 (2) to obtain

$$\begin{aligned} 2\sigma \int \left\{\left(|D^2v|^2 - \beta^2|Dv|^2\right) + \alpha^2\left(|Dv|^2 - \beta^2|v|^2\right)\right\} + \frac{2}{R}\int \left(|D\phi|^2 - \beta^2|\phi|^2\right) &= \\ -\int \left(U''' + (\alpha^2+\beta^2)U'\right)T = \int \left(U'' + (\alpha^2+\beta^2)(U - U_0)\right) T' \end{aligned} \tag{7}$$

The left hand side of (7) is not necessarily positive even for $\sigma > 0$ unless β is small enough. Indeed, the left hand side consists of integrals of the form $\int(|Df|^2 - \beta^2|f|^2)$ and each of these integrals will be positive provided $\beta \leq \pi/2$ for free slip boundary

[1] In the classical literature, *e.g.* [1, Chap. 4], it is common to use a streamfunction $\phi(y)$ for the Rayleigh and Orr-Sommerfeld equations, we prefer to use the vertical velocity v in those equations and our $\phi = (D^2 - \alpha^2)v$ is effectively the vorticity, $\omega = \partial_x v - \partial_y u = (i\alpha)^{-1}\phi(y)e^{\lambda t}e^{i\alpha x}$.

conditions $v = D^2v = \phi = 0$ at the walls at $y = \pm 1$ since for such functions one can show by variational calculus that

$$\int_{-1}^{1} |Dv|^2 dy \geq \frac{\pi^2}{4} \int_{-1}^{1} |v|^2 dy, \tag{8}$$

$$\int_{-1}^{1} |D\phi|^2 dy \geq \frac{\pi^2}{4} \int_{-1}^{1} |\phi|^2 dy, \tag{9}$$

$$\int_{-1}^{1} |D^2v|^2 dy \geq \frac{\pi^2}{4} \int_{-1}^{1} |Dv|^2 dy, \tag{10}$$

so the left hand side of (7) will always be positive if $\sigma > 0$ (instability) and $\beta^2 \leq \pi^2/4$. This yields another necessary condition for instability

$$-\int \left(U''' + \left(\alpha^2 + \frac{\pi^2}{4}\right) U'\right) T = \int \left(U'' + \left(\alpha^2 + \frac{\pi^2}{4}\right)(U - U_0)\right) T' \geq 0. \tag{11}$$

Condition (11) is expressed for a domain normalized to $-1 \leq y \leq 1$, for $y_1 \leq y \leq y_2$ the factor $\pi^2/4$ should be replaced by π^2/H^2, with $H = y_2 - y_1$.

Thus for a linear shear flow instability, $\sigma > 0$, we must have positive energy production $\int U'T = -\int (U - U_0)T' > 0$ from (2), always, as well as condition (11) for *viscous flow with free-slip* perturbations. Note that (11) together with (2) includes and therefore supersedes (3) and (6) and we obtain the necessary conditions for linear instability for *viscous flow with free-slip*, or for *inviscid flow*,

$$\boxed{\int (-U''')\, T \;\geq \left(\alpha^2 + \frac{\pi^2}{H^2}\right) \int U'\, T \;> 0} \tag{12}$$

which, after integration by parts with $T = 0$ at the walls, can also be written as

$$\boxed{\int U''T' \;\geq \left(\alpha^2 + \frac{\pi^2}{H^2}\right) \int (U_0 - U)\, T' \;> 0} \tag{13}$$

where $H = y_2 - y_1$ is the total channel height, U_0 is an arbitrary constant, $\int \equiv \int_{y_1}^{y_2} dy$ and $T = T(y)$ is the perturbation Reynolds stress (1).

1.1 Linear stability of Couette, Poiseuille and Kolmogorov

The enstrophy equation (3) allows us to conclude that plane Couette flow $U = y$, plane Poiseuille flow $U = 1 - y^2$ and any combination of Couette and Poiseuille $U = a + by + cy^2$ for any constant a, b, c (*i.e.* shear flow driven by both a pressure gradient and imposed stress at the walls) are *linearly stable* for *inviscid or viscous flow with free-slip*, since all these flows have $U''' = 0$ and no enstrophy production, therefore $\sigma < 0$ for any $0 \leq R < \infty$ from (3).[2]

Condition (12), or (13), allows us to show linear stability for *free-slip* of the *Kolmogorov flow*

$$U(y) = \frac{\sin(\beta y)}{\sin \beta} \tag{14}$$

[2]We stress again that these results only apply to viscous flows with *stress* boundary conditions, that is, $\mathbf{v} \cdot \mathbf{n}$ and $\partial_n \mathbf{v}_\parallel$ fixed (*i.e.* fixed stress $\nu \partial_n \mathbf{v}_\parallel$), where $\mathbf{n}$ is the unit normal to the wall and $\mathbf{v}_\parallel = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ is parallel to the wall. From incompressibility, this yields $v = D^2v = 0$ at the walls. For no-slip, (3) has an extra boundary term of indefinite sign.

whenever $\beta \leq \pi/2$. The Kolmorogov flow (14) is normalized so $U(\pm 1) = \pm 1$ as for Couette flow which it asymptotes to for $\beta \to 0$, and we can take $\beta \geq 0$ without loss of generality. The Kolmogorov flow is an *inflectional profile* with a vorticity maximum at $y = 0$. For (14), we have $-U''' = \beta^2 U' = \beta^3 \cos(\beta y)/\sin\beta$ so instability of the *wall-bounded* Kolmogorov flow requires, from (12), that

$$\int_{-1}^{1} \cos(\beta y)\, T\, dy > 0 \qquad \text{and} \qquad \beta^2 \geq \alpha^2 + \frac{\pi^2}{4}, \tag{15}$$

where $T(y)$ is defined in (1), so Kolmogorov flow (14) with $0 \leq \beta \leq \pi/2$ is linearly stable. This includes Couette flow for $\beta \to 0$ and the flow $U(y) = \sin(\pi y/2)$ used in the derivation of the SSP model [9], as well as all sinusoidal profiles between those two flows.

Lou Howard (1997, private communication) provided a proof for the linear stability of the $U = \sin(\pi y/2)$ viscous flow with free-slip perturbations used in [9]. His proof made use of the energy (2) and enstrophy (3) equations and $U'' = (-\pi^2/4)U$ for $U = \sin \pi y/2$ to eliminate the production terms through the combination (3) - $\pi^2/4$ (2).

Linear instability for *inviscid* or *viscous* flow with *no-slip or free-slip* requires positive perturbation energy production $\int U'T > 0$ from (2). If we could show that $U'T \geq 0$ *pointwise*, for instance, then we could generalize the classic Rayleigh and Fjortoft theorems of inviscid flow (see below) to *viscous* flow with *free-slip* perturbations. Indeed if we *assume* that $U'T$ is positive pointwise, not just on average as required by (12), then

$$\int (-U''')T' = \int \left(\frac{-U'''}{U'}\right) U'T \;\leq \max_y \left(\frac{-U'''}{U'}\right) \int U'T$$

and (12) would yield the necessary condition[3]

$$\max_{y_1 \leq y \leq y_2} \left(\frac{-U'''}{U'}\right) \geq \alpha^2 + \pi^2/H^2. \tag{16}$$

This would be a stronger version of Fjortoft's theorem, implying for instance that a flow such as $U = y^3$ with $U' = 3y^2 \geq 0$ and $U''' = 6 > 0$ could not be unstable but other inflectional flows such as $U = \tanh(\beta y)$ could be unstable provided β is large enough. This would be a nice result since the linear stability of shear flows (see *e.g.* [1, Chap. 4]) is in an unsatisfactory state of affairs, with classic inflectional instability results derived only for *inviscid* flows. If we could extend those results to *viscous* flow with *free-slip*, this would certify that the difference is not between inviscid or viscous flow, but between free-slip or no-slip, as numerical calculations indicate. The physical difference arising because no-slip allows the generation of enstrophy at the walls but free-slip or viscosity-free do not.

1.2 Inviscid results

The energy (2) and enstrophy (3) integrals, and the instability conditions (12), (13), still apply for *inviscid flow* with $1/R \equiv 0$ in which case the *Orr-Sommerfeld equation* for $v(y)$

$$\left[\lambda + i\alpha U - \frac{1}{R}(D^2 - \alpha^2)\right](D^2 - \alpha^2)v - U'' i\alpha v = 0. \tag{17}$$

[3] as claimed in the lecture but flagged by Matt Chantry as only valid for $U'T \geq 0$ pointwise, which we have not shown. Good eye, Matt!

reduces to the *Rayleigh equation*

$$(U-c)(D^2-\alpha^2)v - U''v = 0 \tag{18}$$

with $\lambda \triangleq -i\alpha c$, that is $c = i\lambda/\alpha = \omega/\alpha + i\sigma/\alpha$ for $\lambda = \sigma - i\omega$ with $i^2 = -1$ and σ, ω real. So an instability for Rayleigh's equation occurs when $\Im(c) = \sigma/\alpha > 0$, taking $\alpha > 0$ without loss of generality. The only boundary condition for Rayleigh's equation is no-flow through the walls, that is $v = 0$ at $y = y_1$ and y_2.

Rayleigh's equation (18) allows us to derive an expression for T', the Reynolds force. Substituting for D^2v from (18) into $T' \equiv DT = dT/dy$ calculated from (1) gives

$$T' = i\alpha(vD^2v^* - v^*D^2v) = 2\sigma\,\frac{U''}{|U-c|^2}\,|v|^2 \tag{19}$$

hence T' has the sign of U'' when $\sigma = \Re(\lambda) > 0$ (instability).

Rayleigh's theorem (1880). Since $T = 0$ at the walls (1), T' and therefore U'' *must change sign in the domain for instability.* Thus U'' must vanish somewhere in the domain but not everywhere (Couette flow) since $\sigma = 0$ from (2) and (19) when $1/R \equiv 0$ and $U'' = 0$ everywhere.

Fjortoft's theorem (1950). Substituting for T' from (19) into (2) shows that instability requires $(U_0 - U)U'' \geq 0$ somewhere in the domain, for any U_0, which again gives Rayleigh's theorem that U'' must change sign in the domain. Picking $U_0 = U(y_s) = U_s$ where $U''(y_s) = 0$, so both $U_s - U$ and U'' change sign when y crosses y_s, gives the perturbation energy equation (2) as

$$\int_{y_1}^{y_2} \left(|Dv|^2 + \alpha^2|v|^2\right)\,dy = \int_{y_1}^{y_2} \frac{(U_s-U)U''}{|U-c|^2}\,|v|^2\,dy \;\geq 0, \tag{20}$$

for $\sigma \neq 0$ and *Fjortoft's theorem* that $(U_s - U)U'' \geq 0$ somewhere in the domain is necessary for instability. This implies linear stability of flows such as $U = y^3$ for which $(U_s - U)U'' = -6y^4 \leq 0$, but possible instability of flows such as $U = \sin(\beta y)/\sin\beta$ that have $U'' = \beta^2(U_s - U)$, see *e.g.* [1, Fig. 4.2].

We can go further by substituting for T' from (19) into (11) or (13), with $U_0 = U_s$, to find that $(U'')^2 \geq (\alpha^2 + \pi^2/H^2)(U_s - U)U''$ somewhere in the domain. If we now *assume* that $(U_s - U)U'' \geq 0$ everywhere, we obtain that

$$\max_{y_1\leq y\leq y_2}\left(\frac{U''}{U_s-U}\right) \geq \alpha^2 + \frac{\pi^2}{H^2} \tag{21}$$

is necessary for instabilitym where H is the full height of the channel. This implies stability of the Kolmorogov flows $U = \sin(\beta y)/\sin\beta$ when $|\beta| \leq \pi/H$ as we already established for viscous flow with free-slip, but now also includes other similar flows such as $U = \tanh\beta y$ which are only unstable for β large enough (left to the reader). Condition (21) effectively contains the results of Friedrichs (1942) and Drazin and Howard (1966) [1, p. 133, 134]. Our derivation is more straightforward but the result is not quite identical since Friedrichs provides an expression for a neutral wavenumber. Condition (21) shows that inflectional instabilities are larger scale instabilities, that is, they require $0 \leq \alpha^2 \leq \beta_s^2 - \pi^2/H^2$, where $\beta_s^2 \equiv \max(U''/(U_s - U))$.

Since the production integral on the right hand side of (20) can written for any constant U_0 in place of U_s and in particular for $U_0 = c_r = \Re(c)$, and since $|U - c|^2 = (c_r - U)^2 + c_i^2$ with $c_i = \Im(c) = \sigma/\alpha$, we can infer that while $U'' = 0$ somewhere is necessary for instability, the maximum instability (max c_i) occurs for values of c_r that tend to maximize $U''/(c_r - U)$ and functions $v(y)$ that are largest near those maxima. For profiles that are anti-symmetric about the inflection point, such as $U = \sin\beta y$ or $\tanh\beta y$, this will likely be for $c_r = U_s$.

2 Rayleigh's piecewise linear models

Rayleigh's eigenvalue problem (18) is difficult to solve when $U(y)$ is a smoothly varying function (figure 1(a)). However, if $U(y)$ is defined as a piecewise linear function (as shown in figure 1(b)), then the solutions of Rayleigh's equation are simple exponential or hyperbolic functions which must satisfy certain matching conditions at a discontinuity of $U(y)$ or $U'(y)$ [1].

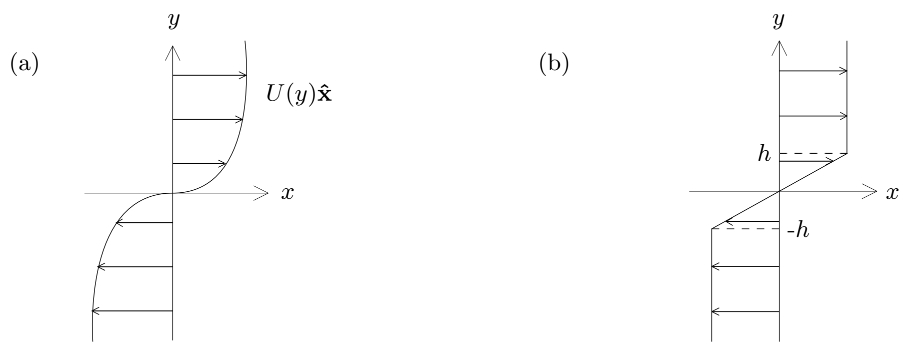

Figure 1: (a) Unbounded smooth shear flow. (b) Piecewise-linear unbounded shear flow.

The matching conditions can be derived by going back to the primitive equations [1, §23] and [8, §6.2.1] and the reader should study those derivations. Here, we start from Rayleigh's equation (18) and imagine a continuous deformation from a smooth profile to a piecewise linear profile, for instance a continuous deformation of $U = \tanh(y/h)$ into the piecewise linear profile in fig. 1(b)). Then Rayleigh's equation applies but $U'' \to \infty$ at corners and 0 everywhere else, *i.e.* U'' tends to a sum of Dirac delta functions and Rayleigh's equation implies that $D^2 v \to \infty$ at those points also, to balance the U'' divergences. That is, the jumps in U' must be balanced by jumps in v' as governed by Rayleigh's equation. Indeed, Rayleigh's equation (18) can be rewritten in the form

$$\big((U-c)v' - U'v\big)' - \alpha^2 (U-c)v = 0, \tag{22}$$

and integrating (22) across a vanishing rapid transition region for U', say from $y = y_0 - \epsilon$ to $y = y_0 + \epsilon$ with $\epsilon \to 0^+$ gives the jump condition

$$\Big[(U-c)v' - U'v\Big]_{y_0^-}^{y_0^+} = 0. \tag{23}$$

This jump condition corresponds to continuity of pressure [1, §23] and [8, §6.2.1].

If we also allow for jumps in U, these must be matched by jumps in v and that balance is revealed by rewriting (22) as

$$\left((U-c)^2 \left(\frac{v}{U-c}\right)'\right)' - \alpha^2 (U-c)v = 0. \tag{24}$$

which shows that $v/(U-c)$ cannot jump since such a jump could not be balanced in Rayleigh's equation. Thus, the jump conditions for v at a jump of U is

$$\left[\frac{v}{U-c}\right]_{y_0^-}^{y_0^+} = 0. \tag{25}$$

A discontinuous U profile corresponds to the Kelvin-Helmholtz model with a sharp interface between two differentially moving fluid layers. The jump condition (25) corresponds to the linearized material interface condition [1, §23] and [8, §6.2.1].

Away from jumps, when the velocity profile is piecewise linear, $U'' = 0$, and so Rayleigh's stability equation (18) has the general solution

$$v(y) = Ae^{\alpha y} + Be^{-\alpha y} \tag{26}$$

for arbitrary constants A,B. Therefore, we can use conditions (23) and (25) to match solutions of the form (26) to solve any problem with a piecewise linear velocity profile.

For the piecewise linear unbounded shear flow, we take

$$U(y) = \begin{cases} U_0 & \text{if } y \geq h, \\ U_0 y/h & \text{if } -h \leq y \leq h, \\ -U_0 & \text{if } y \leq -h, \end{cases} \tag{27}$$

as in figure 1b. Note that $U'' = (-U_0/h)\delta(y-h) + (U_0/h)\delta(y+h)$ where $\delta(\cdot)$ is the Dirac delta function and that U'' *changes sign*. We could consider this problem as the limit for $\epsilon \to 0^+$ of the smooth profile with $U'' = (-U_0/h)G(y-h,\epsilon)+(U_0/h)G(y+h,\epsilon)$ where $G(y,\epsilon) = (\pi\epsilon)^{-1/2}\exp(-y^2/\epsilon)$ is the standard Gaussian.

Solving (18) for (27) with $v \to 0$ as $y \to \pm\infty$ gives

$$v(y) = \begin{cases} Ae^{-\alpha(y-h)} & \text{if } y > h, \\ Be^{\alpha y} + Ce^{-\alpha y} & \text{if } -h < y < h, \\ De^{\alpha(y+h)} & \text{if } y > -h, \end{cases} \tag{28}$$

with $\alpha > 0$ (and D here is a constant not the d/dy shorthand as before). Since U is continuous, the jump condition (25) reduces to continuity of v at $y = \pm h$, hence

$$\begin{aligned} A &= Be^{\alpha h} + Ce^{-\alpha h}, \\ D &= Be^{-\alpha h} + Ce^{\alpha h}. \end{aligned} \tag{29}$$

It is now convenient to let

$$\hat{c} = \frac{c}{U_0}, \quad \hat{\alpha} = \alpha h, \tag{30}$$

(or equivalently taking h and U_0 has length and velocity scales leading to $h \equiv 1$ and $U_0 \equiv 1$) then applying the jump condition (23) at $y = \pm h$, substituting for A and D from (29) gives

$$\begin{aligned} \big(2\hat{\alpha}(1-\hat{c}) - 1\big)Ce^{\hat{\alpha}} &= Be^{-\hat{\alpha}}, \\ \big(2\hat{\alpha}(1+\hat{c}) - 1\big)Be^{\hat{\alpha}} &= Ce^{-\hat{\alpha}}, \end{aligned} \tag{31}$$

which after elimination of B and C yields $\big((2\hat{\alpha}-1)^2 - 4\hat{\alpha}^2\hat{c}^2\big) = e^{-4\hat{\alpha}}$ and

$$\hat{c}^2 = \frac{(1-2\hat{\alpha})^2 - e^{-4\hat{\alpha}}}{4\hat{\alpha}^2}, \tag{32}$$

such that $\hat{c}^2 \to -1$ as $\hat{\alpha} \to 0$, $\hat{c}^2 = 0$ at $\hat{\alpha} \approx 0.63$ and $\hat{c}^2 < 0$ in $0 < \hat{\alpha} \lesssim 0.63$. A negative $\hat{c}^2$ means $c = c_r + ic_i$ with $c_r = 0$ and $c_i = \pm|c|$, hence instability. The growth rate (18) $\lambda = -i\alpha c = \alpha c_i$ is real when c^2 is negative. Define $\hat{\lambda} = \hat{\alpha}\hat{c}_i = \alpha c_i(h/U_0)$, so $\hat{\lambda} = \lambda(h/U_0)$ and this non-dimensional growth rate is plotted in fig. 2 as a function of the non-dimensional wavenumber $\hat{\alpha} = \alpha h$.

Kelvin-Helmholtz. The limit $h \to 0$ yields the Kelvin-Helmholtz model with $\hat{\alpha} = \alpha h \to 0$ in (32) yielding $\hat{c}^2 \to -1$ so $c = \pm i U_0$ and $\lambda = -i\alpha c = \pm \alpha U_0$. The Kelvin-Helmholtz model is *ill-posed* since $\lambda = \alpha U_0$ can be as large as one desires by taking α large enough, but Rayleigh's piecewise linear model with a length scale h, eqns. (27), (32) and fig. 2, is well-posed and gives a qualitatively and quantitatively valid picture of the instability that only occurs for $\alpha \lesssim 0.63/h$. Although, 'Kelvin-Helmholtz instability' is often used to describe general inflectional instability and vortex roll-up, the Kelvin-Helmholtz model is a bit too singular to provide insights into the instability for smooth profiles $U(y)$. The Rayleigh model (27) is more physical and shows that the instability results from the interaction between two regions where $U''/(c_r - U)$ is large and positive with U'' of opposite signs. Valis [8, §6.2.4] provides a useful interpretation of the instability as an interaction between edge waves for the Rayleigh model.

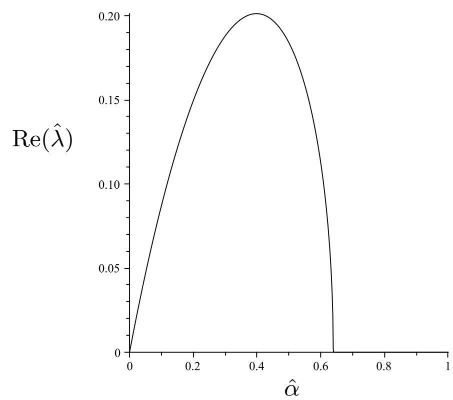

Figure 2: Growth rate $\hat{\lambda} = \hat{\alpha}\hat{c}_i = \alpha c_i (h/U_0)$ with c given by equation (32). The flow is unstable for $\hat{\alpha} = \alpha h < 0.63$.

Reynolds stress. The perturbation Reynolds stress is given by (1)

$$-\overline{uv} = \frac{i}{\alpha}\left(v\frac{dv^*}{dy} - v^*\frac{dv}{dy}\right). \tag{33}$$

In the case of piecewise linear unbounded shear flow (28) this gives

$$-\overline{uv} = \begin{cases} 0 & \text{if } y > h, \\ 2i(BC^* - B^*C) & \text{if } -h < y < h, \\ 0 & \text{if } y > -h. \end{cases} \tag{34}$$

then eliminating C using (31) with $c = ic_i$ and $\hat{\sigma} = \hat{\alpha}\hat{c}_i > 0$ gives

$$-\overline{uv} = 4\hat{\sigma}|B|^2 e^{2\hat{\alpha}} > 0 \tag{35}$$

in $-h < y < h$, where $\hat{\alpha} = \alpha h$ and $\hat{\sigma} = \sigma(h/U_0)$ with $\sigma = \alpha c_i > 0$ for an unstable mode. Therefore, constant positive perturbation Reynolds stress $-\overline{uv}$ occurs throughout the shear layer and $U'T \geq 0$ pointwise (2) (but this is for the inviscid problem). The Reynolds stress $-\overline{uv}$ transports momentum from $y = h$ to $y = -h$ and vice-versa. The Reynolds *force* onto the mean flow $-d\overline{uv}/dy$ consists of two delta functions, one negative at $y = h$ and a positive at $y = -h$, slowing down the mean at $y = h$ and speeding it up at $y = -h$.

3 Instability from viscosity and no-slip

Remarkably, viscosity and *no-slip* at the walls can lead to linear instability even for flows with $U''' = 0$, such as plane Poiseuille flow $U = 1 - y^2$, that are stable for free-slip as shown in sect. 1. In plane Poiseuille flow, Heisenberg [2] found a weak linear 2D instability, that occurs at $R \simeq 5772$ [4] and disappears as $R \to \infty$. In boundary layer flows Tollmien [7] and Schlichting [6] demonstrated a weak 2D instability which has a critical Reynolds number of approximately $R \simeq 500$ and again disappears as $R \to \infty$. However, unlike the previous two flows, Romanov [5] proved that plane Couette flow is linearly stable for all values of R (although this was already believed since the work of Hopf (1914), [1, §31.1]). While pipe flow (or Hagen-Poiseuille flow) has not been proven linearly stable for all R, it is believed to be so, and this has been shown up to $Re \simeq 10^5$ experimentally and $R \simeq 10^7$ computationally (see lecture 1). When the no-slip boundary conditions are replaced by free-slip boundary conditions for the perturbations then we showed earlier in sect. 1 that plane Poiseuille and Couette flows are linearly stable for all R. The instability for viscous flow with no-slip in channel flow arises because of the generation of vorticity at the boundary (4). This is a delicate process because viscosity leads to dissipation of enstrophy in the bulk as well as generation of enstrophy at the boundary (3), these two viscous effects are of the same order and oppose each other.

(Note: A lecture on the Orr-Sommerfeld equation for $R < \infty$, with a look at Heisenberg and Tollmien's work and critical layers was skipped in the GFD program.)

4 Failures of linear theories

We now summarise the results derived from linear theory in the last two lectures. From the previous lecture we have the governing linear equations

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right)\eta = -U'\frac{\partial v}{\partial z}, \tag{36}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{Re}\nabla^2\right)\nabla^2 v - U''\frac{\partial v}{\partial x} = 0, \tag{37}$$

where $v = \hat{\mathbf{y}} \cdot \mathbf{v}$ and $\eta = \hat{\mathbf{y}} \cdot \boldsymbol{\nabla} \times \mathbf{v} = \partial_z u - \partial_x w$. For $v = 0$, we can show that $\eta \to 0$ for $\eta = 0$ or $\partial_n \eta = 0$ at the walls, since the homogeneous η equation is an advection diffusion equation. Exponential instabilities therefore can only originate from the v equation and Squire's theorem (lecture 3) shows that 2D (x, y), that is independent of the spanwise direction z, are more unstable than 3D disturbances. However, the canonical shear flows (Couette, Poiseuille, pipe) do not have a linear instability, except for a weak linear instability for *viscous* plane Poiseuille flow with *no slip* at the walls.

Energy stability on the other hand (lecture 3 and [1, §53.1]) shows that 2D perturbations depending on (y, z) only, independent of the streamwise direction x, lead to the lowest Reynolds number below which the flow is absolutely stable. Hence linear stability theory and energy stability theory give, literally, *orthogonal* results!

The x-independent perturbations of energy stability theory lead to the largest initial perturbation energy growth since they maximize production over dissipation (lecture 3), but such x independent perturbations ultimately decay. We discussed this in lecture 1 and can show it by considering the *full* non-linear Navier-Stokes equations with no

x-dependence,

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \cancelto{0}{u\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\cancelto{0}{\frac{\partial p}{\partial x}} + F + \frac{1}{R}\nabla^2 u, \\
\frac{\partial v}{\partial t} + \cancelto{0}{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \frac{1}{R}\nabla^2 v, \\
\frac{\partial w}{\partial t} + \cancelto{0}{u\frac{\partial w}{\partial x}} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w,
\end{aligned}
\tag{38}
$$

where $\mathbf{v} = (u, v, w)$ and where F in the u equation is a driving body force. The continuity equation reduces to $\boldsymbol{\nabla}\cdot\mathbf{v} = \partial_y v + \partial_z w = 0$. Hence, the equations for v and w decouple from the equation for u and the latter is a *passive scalar* forced by F and redistributed by v, w. This decoupling implies that v and w do not have any forcing and therefore they decay because of viscosity, no matter their initial amplitude [3]. This was discussed and proved in lecture 1. The proof is simple and consists in deriving the equation for the cross-stream kinetic energy $\int_A (v^2 + w^2)$ where A is the cross-section and showing that $\frac{d}{dt}\int_A (v^2 + w^2) = -(1/R)\int_A \left(|\boldsymbol{\nabla} v|^2 + |\boldsymbol{\nabla} w|^2\right) \le 0$.

These x-independent perturbations also lead to the largest linear growth of the perturbation energy. For such perturbations, $\eta = \partial_z u - \partial_x w$ reduces to

$$
\eta = \frac{\partial u}{\partial z},
$$

and we can therefore integrate equation (36) with respect to z to recover the streamwise $u = U(y) + \tilde{u}$ velocity equation (38) linearized about the base shear flow $U(y)$

$$
\frac{\partial \tilde{u}}{\partial t} - \frac{1}{R}\nabla^2 \tilde{u} = -vU'. \tag{39}
$$

Hence, x-independent but z-dependent $v(y, z)$ perturbations can generate large perturbations of streamwise velocity u and large $\eta = \partial_z u$. However, they eventually decay in the linear theory *as well as* in the full x-independent nonlinear theory, as there is no feedback upon v. The reader is referred to the discussions and models in lecture 1.

5 3D, nonlinear 'instability'

Thus linear theory of shear flows fails. Energy stability and upper bound theories (lecture 2) suggest x-independent perturbations as most effective at initial perturbation energy growth and maximum momentum transport and energy dissipation, but truly x-independent perturbations always decay, for all amplitudes. So we need a *nonlinear, 3D* theory. Ouch!

Yet, we're not far. The x-independent perturbations indeed are very good at redistributing the streamwise velocity u and transporting momentum, that is maximizing $-\overline{uv}$ and perturbation energy production $-\overline{uv}\,U'$. This is clear from equation (39) which for large R gives $\tilde{u} \sim -vU't$ so $-\overline{uv} \sim \overline{v^2}\,U't$.

These perturbations are necessarily spanwise z dependent, otherwise continuity and the boundary conditions would require $v = 0$. These perturbations typically introduce z-inflections in the streamwise velocity profile and those lead to instabilities of inflectional type, but as a result of z inflections, not y inflections as in the classical linear theory. These inflectional instabilities extract energy and momentum from the u-fluctuations of course and will therefore *accelerate* the return to the laminar flow, *unless* they manage to regenerate v. This seems like a lot of ifs, however that is essentially the fundamental *self-sustaining process* that leads to the possibility of 3D, nonlinear states disconnected from the laminar flow, and ultimately the sustenance of turbulent shear flows. The self-sustaining process will be written up in more detail in the next lectures. (There was lots of hand-waving and jumping around by the lecturer that is difficult to write-up).

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