1. [10 pts] Evaluate the following limits, showing the main steps in your reasoning.

$$\lim_{x\to 0} \frac{\sin x - x}{\sin x^3} \tag{1}$$

$= 0/0$ so we can use l'Hospital's rule. Or we can use Taylor's formula for $\sin\theta = \theta - \theta^3/6 + O(\theta^5)$ and

$$\lim_{x\to 0} \frac{\sin x - x}{\sin x^3} = \lim_{x\to 0} \frac{(x - x^3/6 + O(x^5)) - x}{x^3 + O(x^9)} = \lim_{x\to 0} \frac{(-x^3/6 + O(x^5))}{x^3 + O(x^9)} = -1/6.$$

Reminder about meaning of the "big Oh" notation:

$$O(x^5) \text{ as } x \to 0 \iff \lim_{x\to 0} \frac{O(x^5)}{x^5} = C < \infty.$$

In other words $O(x^5) \approx Cx^5$ for some finite C as $x \to 0$.

$$\lim_{\theta\to 0} \frac{\tan\theta - \theta}{\cos\theta^3} = 0/1 = 0. \tag{2}$$

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2. [20pts] Sketch the curves (a) $y = x^2/(1 - x^3)$ and (b) $y = 1/x - 1/\sin x$. Identify the exact (not numerical approximations) extrema and inflection points in case (a).

Ask your TA for more details and a nice sketch if these hints are not enough for you. Remember we want a good sketch!

(a) $f(x) = x^2/(1-x^3)$ has a singularity at $x = 1$ with $\lim_{x\to 1^-} f(x) = +\infty$ and $\lim_{x\to 1^+} f(x) = -\infty$; $f(x) \approx x^2$ for small x; $f(x) \approx -1/x$ for large $|x|$. This gives us a very good idea of what the plot looks like, but it's up to you to plot it.
For verification/refinement, we compute

$$f'(x) = \frac{x(2 + x^3)}{(1 - x^3)^2}, \qquad f''(x) = \frac{2(1 + 7x^3 + x^6)}{(1 - x^3)^3}$$

So the function is increasing ($f' > 0$) if $x > 0$. $f' = 0$ at $x = 0$ and at $x = -2^{1/3}$. It is decreasing ($f' < 0$) if $-2^{1/3} < x < 0$ and increasing if $x < -2^{1/3}$. So $x = 0$ is a local min and $x = -2^{1/3}$ is a local max. We expect two points of inflection and indeed $f'' = 0$ at two $x's$: the two roots of $y^2 + 7y + 1 = 0$ with $y = x^3$,...

(b) $f(x) = 1/x - 1/\sin x$ has lots of singularities! $f(x)$ is singular whenever $\sin x = 0$ i.e. $x = n\pi$, where n is any integer. But there is no singularity at $x = 0$ because the $1/x$ and the $1/\sin x$ balance perfectly there. We studied that limit in class (see example 7 in 3.9), it is zero. More precisely,

$$f(x) = \frac{\sin x - x}{x\sin x} \approx \frac{(x - x^3/6 + \cdots) - x}{x(x - x^3/6 + \cdots)} \approx -x/6,$$

so $f(x)$ looks like $-x/6$ near $x = 0$. But it's ony at zero that $x = \sin x$, they are never equal for any other x, so $1/x$ will never be equal to $1/\sin x$ either, except in the limit $\to 0$ where they balance out perfectly.
We're not looking for a very accurate sketch here, just a good idea of what this beast looks like, and that turns out to be largely determined by the left and right limits at the special points $x = n\pi$. For $n \neq 0$ the $1/x$ doesn't do much, it's the $-1/\sin x$ we need to worry about because

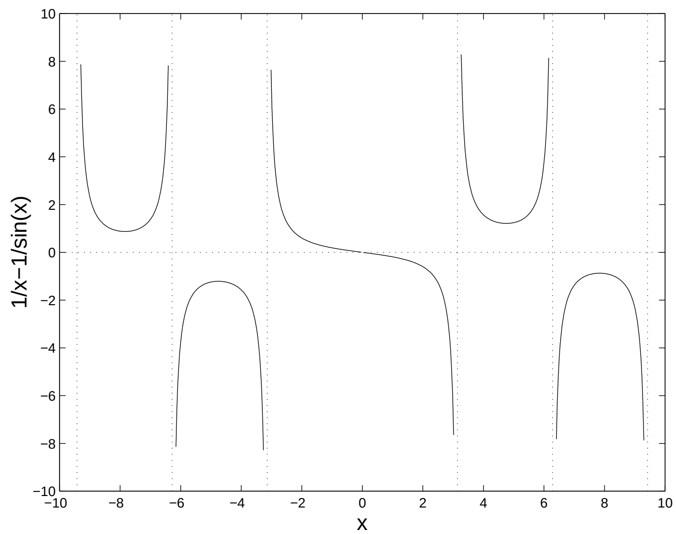
$$\lim_{x\to n\pi} \frac{-1}{\sin x} = \pm\infty.$$

It's probably easiest to figure out all those limits by using the unit circle carefully (plot it!).

$$\lim_{x\to\pi^-}\frac{-1}{\sin x} = -\infty, \quad \lim_{x\to\pi^+}\frac{-1}{\sin x} = +\infty,$$

$$\lim_{x\to 2\pi^-}\frac{-1}{\sin x} = +\infty, \quad \lim_{x\to 2\pi^+}\frac{-1}{\sin x} = -\infty,$$

Those are enough to figure out all the limits because $\sin(x \pm 2\pi) = \sin x$ as is obvious from the unit circle.



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3. [20pts] Calculate the following

$$\int x^2\cos(x^3+1)dx = \int \cos u\ du/3 = (\sin u)/3 = \frac{\sin(x^3+1)}{3} \tag{3}$$

This is a fundamental one that you need to know, see exam 1

$$\int \frac{d\theta}{\cos^2\theta} = \tan\theta \tag{4}$$

$$\int x^2\cos x dx \equiv \int fg'dx = fg - \int f'gdx \equiv x^2\sin x - 2\int x\sin x dx \tag{5}$$

but

$$\int x\sin x dx \equiv \int uv'dx = uv - \int u'vdx \equiv -x\cos x + \int \cos x dx = -x\cos x + \sin x,$$

so

$$\int x^2\cos x dx = x^2\sin x + 2x\cos x - 2\sin x.$$

Check it!

$$\int \frac{u}{\sqrt{1-4u^2}}du = \int \frac{1}{\sqrt{v}}\frac{dv}{(-8)} = -\frac{1}{4}\sqrt{v} = -\frac{1}{4}\sqrt{1-4u^2} \tag{6}$$

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4. [10pts] The volume of a cylindrical can of soda is 355 cubic centimeters. The can is made of aluminum of fixed thickness. Find the dimensions of the can that requires the least amount of aluminum.

$V = \pi r^2 h$, $A = 2\pi r h + 2\pi r^2$. V is fixed, say $V = V_0$, A is not. We want to min A. Since V is fixed, r and h are not independent, indeed $h = V_0/(\pi r^2)$ and $A = A(r) = 2\pi r^2 + 2V_0/r$. Quick check: A is an area, r^2 is an area, V_0/r is a volume divided by a length, so it too has dimensions of area. All looks good. $A(r) \approx 2V_0/r \to +\infty$ as $r \to 0$ and $A(r) \approx \pi r^2$ for large r. Clearly there is a minimum for some finite positive r. Find it using $A' = 0$. $A'(r) = 4\pi r - 2V_0/r^2$, so the min is achieved when $r^3 = V_0/(2\pi)$, i.e. when $r = h/2$.

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5. [10pts] A spotlight on the ground shines on a wall 12m away. A player from the UW women's volleyball team who is 2m tall is walking toward the wall at 8/5 m/sec. How fast is her shadow on the wall changing when she is 4 m from the wall?

Make a good sketch. Let's call x the distance from the spotlight to the woman, h her height, L the distance from the spotlight to the wall, and S the height of her shadow on the wall. L and h are constants, let's emphasize that by calling them L_0 and h_0, they do not change. S and x are functions of time: $S = S(t)$, $x = x(t)$. Using similar triangles we get $S/L = h/x$ or $S(t)/L_0 = h_0/x(t)$ i.e. $x(t)S(t) = L_0 h_0$. By the product rule

$$\frac{d}{dt}(xS) = \frac{dx}{dt}S + x\frac{dS}{dt} = \frac{d}{dt}(L_0 h_0) = 0,$$

so

$$\frac{dS}{dt} = -\frac{dx}{dt}\frac{S}{x},$$

but $S(t) = h_0 L_0/x(t)$, so

$$\frac{dS}{dt} = -\frac{dx}{dt}\frac{L_0 h_0}{x^2}.$$

Now if $dx/dt = 8/5$ and $x = 12 - 4 = 8$ then $dS/dt = -(8/5)(h_0 L_0)/8^2 = -(8/5)(24/8^2) = -3/5$. We can go further now that we know some differential equations:

$$\frac{dx}{dt} = \frac{8}{5} \Longrightarrow x(t) = \frac{8}{5}t + C.$$

Let's pick $t = 0$ when $x = 8$, so $C = 8$. giving

$$\frac{dS}{dt} = \frac{-15}{(5+t)^2}.$$

We can solve this by separating variables

$$dS = -15\frac{dt}{(5+t)^2} \longrightarrow S = \frac{15}{5+t} + C_2$$

but $S(0) = h_0 L_0/x(0) = 3$ so $C_2 = 0$. That $S(t)$ is only true until she crashes into the wall...

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6. [10pts] How many zeros does the function $f(x) = (x^2 - 2x + 2)/(x - 1)$ have between 0 and 2?

$f(x)$ has a singularity at $x = 1$, so we better watch out! $f(0) = -2$, $f(2) = 2$ but the function is not continuous in [0,2] so the fact that $f(x)$ changes sign between 0 and 2 does NOT mean there is a zero in that interval.

Now

$$f'(x) = \frac{x(x-2)}{(1-x)^2}, \qquad f''(x) = \frac{-2}{(1-x)^3},$$

so there is a local max at 0 and a local min at 2 because $f'(0) = 0$ with $f''(0) = -2 < 0$ (concave down), $f'(2) = 0$ with $f''(2) = +2 > 0$ (concave up). $f(0) = -2$ is a local max, as x increases towards 1, $f(x)$ moves away from 0 and shoots to $-\infty$. As x comes down from the local min at 2, $f(x)$ increases towards $+\infty$. There are NO zeros in [0,2]. Sketch it. It may help also to notice that $f(x) = (x-1) + 1/(x-1)$ and the function $x + 1/x$ is plotted in the book.

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7. [10pts] Solve $dy/dx = 2xy^2$ with $y(1) = C$. Sketch $y(x)$.

$dy/y^2 = 2xdx \longrightarrow -1/y = x^2 + C_1$ or $y(x) = 1/(-C_1 - x^2)$. We need $y(1) = C$ so $y(1) = 1/(-C_1 - 1) = C$ and $C_1 = 1/C - 1$ or

$$y(x) = \frac{C}{C + 1 - Cx^2}.$$

(check: $y(1) = C$, ok). This is an even function, so it is symmetric about $x = 0$, but its shape depends on C. If $(C+1)/C > 0$, meaning $C > 0$ or $C < -1$, then the denominator vanishes for $x = \sqrt{(C+1)/C}$ and $y(x)$ has a singularity there. IF $(C+1)/C$ is negative, i.e. $-1 < C < 0$, then there is no singularity and $y(x) \to 0$ as $x \to \infty$. So there are three types of plots to consider $C < -1$, $C > 0$ and $-1 < C < 0$. There are also two special cases to look at: $C = 0$ for which $y(x) = 0$ and $C = -1$ for which $y(x) = -1/x^2$. You need to understand the meaning of "constant of integration" otherwise this is a (purposefully) confusing exercise! C here is already taken, it has a well defined meaning: $C \equiv y(1)$. Therefore, you need to give another name to the undetermined constant that pops up in the indefinite integration.

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8. [10pts] Show that

$$na^{n-1} \leq \frac{a^n - b^n}{a - b} \leq nb^{n-1}$$

for $n = 1, 2, 3, \ldots$ whenever $b \geq a > 0$.
Reminds you of $(x^n)' = nx^{n-1}$ doesn't it? And the derivative is defined as

$$(x^n)' \equiv \lim_{u \to x} \frac{u^n - x^n}{u - x},$$

isn't it? that looks a lot like the middle term of the inequality.
By the mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $a \leq c \leq b$ (provided $f(x)$ is nice enough in $[a, b]$). The function we need to consider is $f(x) = x^n$, it is more than nice enough for positive integer n, it's positively delightful. So

$$\frac{a^n - b^n}{a - b} = nc^{n-1}$$

with $0 < a \leq c \leq b$, but nx^{n-1} is a function that is *increasing* (or constant) whenever $x > 0$ and $n > 0$, so

$$0 < a \leq c \leq b \Longrightarrow na^{n-1} \leq nc^{n-1} = \frac{a^n - b^n}{a - b} \leq nb^{n-1}.$$