1. [50 pts] Calculate the following expressions. Show your work but NO PARTIAL CREDIT. Full credit only for correct answer with correct derivation.

\( \lim_{a \to \infty} \int_0^a \frac{dx}{1 + x^2} = \lim_{a \to \infty} \arctan x \bigg|_0^a = \lim_{a \to \infty} \arctan a = \pi/2. \)

\( \lim_{\theta \to 0^+} \sqrt{\theta} \ln \theta = \lim_{\theta \to 0^+} \frac{\ln \theta}{\theta^{-1/2}} = 0/0, \) so we can use l'Hospital: \( \lim_{\theta \to 0^+} \frac{1/\theta}{\theta^{-3/2}} = \lim_{\theta \to 0^+} -2\theta^{1/2} = 0. \)

\( \int_{-3}^2 \frac{dx}{\sqrt{17 - x^2}} = \int_{a}^{8/17} \frac{dx}{\sqrt{1 - x^2/17}} \) now let \( u = x/\sqrt{17} \) and the integral is \( \int_{-3/\sqrt{17}}^{2/\sqrt{17}} \frac{du}{\sqrt{1 - u^2}} = \left[ \arcsin u \right]_{-3/\sqrt{17}}^{2/\sqrt{17}} = \arcsin(2/\sqrt{17}) + \arcsin(3/\sqrt{17}). \)

\( \int_2^1 \ln x \, dx = -\int_1^2 d((\ln x)^2/2) = -\left[ (\ln x)^2/2 \right]_1^2 = -0.5(\ln 2)^2. \)

\( \frac{dx^3}{dx} = \frac{d}{dx} \exp(x \ln x) = \exp(x \ln x)(\ln x + 1) \equiv x^2(\ln x + 1). \)

\( \arccos(\cos(7\pi/4)) \) Make a sketch of \( y = \cos x \) to see it. \( \cos 7\pi/4 = \cos(7\pi/4 - 2\pi) = \cos(-\pi/4) = \cos \pi/4. \) By definition, arccos has a range of \( [0, \pi] \) so \( \arccos(\cos(7\pi/4)) = \pi/4. \)

\( \sin(\arccos(1/3)). \) Let \( \alpha = \arccos(1/3), \) then \( \cos \alpha = 1/3 \) and by Pythagoras \( \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha} = \pm \sqrt{8/9} = \pm 2\sqrt{2}/3. \) Which sign is it, \( \pm? \) The arccos is an angle in \( [0, \pi] \) but such angles have positive sines (draw unit circle), so we need to select the \( + \) sign.

\( \ln(x^3e^{-x^2}) = 3 \ln x - x^2. \)

\( \lim_{x \to +\infty} \frac{x^a}{a^x}, \forall a > 0, \) This limit is \( = +\infty \) if \( a \leq 1 \) and \( = +\infty / +\infty \) if \( a > 1. \) We can use l'Hospital for that last case: limit = \( \lim_{x \to +\infty} \frac{ax^{a-1}}{a^x \ln a} = 0 \) if \( a < 2 \) otherwise \( = +\infty / +\infty \), so use l'Hospital again to find that the limit is zero if \( a < 3 \) otherwise \( +\infty / +\infty, \) etc... So the limit is zero if \( a > 1. \)

2. [10pts] Calculate (a) the area between the curves \( y^2 = 4ax \) and \( y^2 = 8ax - 4a^2, \) (b) the volume generated by the rotation of that area about the \( x = 2a \) axis.

Here’s a sketch of the two curves in non-dimensional variables \( x/a, y/a: \)

The curves intersect at \( y^2 = 4ax = 8ax - 4a^2 \implies x = a, y = 2a. \)

(a) Enclosed area = \( \int_{-2a}^{2a} (x_r(y) - x_l(y)) \, dy \) where \( x_r(y) \) is the rightmost curve and \( x_l(y) \) the leftmost, so \( x_r = (y^2 + 4a^2)/(8a) \) and \( x_l(y) = y^2/(4a). \) The area is \( = \int_{-2a}^{2a} (a/2 - y^2/(8a)) \, dy = 4a^2/3. \)

(b) Volume by washers: \( V = \int_{-2a}^{2a} \pi(r_0^2 - r_1^2) \, dy \) where \( r_0 = 2a - x_1 = 2a - y^2/(4a) \) is the outer washer radius and \( r_1 = 2a - x_r = 2a - (y^2 + 4a^2)/(8a) \) is the inner washer radius. There is only straightforward algebra left to do.
3. [10pts] A bowl in the shape of a paraboloid is filled with water at the constant rate of $Q \text{ m}^3/\text{sec}$. How fast is the height of water in the bowl rising?

$y = x^2$ is a parabola but what do $x$ and $y$ mean in that equation? If $x$ is a length then $y = x^2$ is a length squared! (area). We encountered this issue in lecture on Fri Nov 16, 2001. A paraboloid of circular base of radius $R$ and of height $H$ is described by the equation $y = Hx^2/R^2$ which is dimensionally correct, $x$ is a length and $y$ is a length.

If the paraboloid is filled with water up to level $h \neq H$ then the volume of water (by slices) is $V = \int_0^h \pi x^2 dy$, where $y$ is the height of a slice, $dy$ its thickness and $x$ its radius. Now $y = Hx^2/R^2$ so $V = \int_0^h \pi R^2 y/H dy = \pi R^2 h^2/(2H)$ which is indeed a volume.

Then $\frac{dV}{dt} = \frac{1}{H} \frac{dh}{dt} = Q$ or $\frac{dh}{dt} = \frac{QH}{\pi R^2 h}$.

The units on the right-hand side are indeed m/sec. We can solve that differential equation for $h(t)$: $h dh = C dt \rightarrow h^2 = 2Ct + D$ where $C = QH/(\pi R^2)$ and $D$ is an arbitrary constant. If we take $t = 0$ to be the time when there was no water in the bowl, then $D = 0$ and $h = \sqrt{2QHt/(\pi R^2)}$.

4. [10pts] A radioactive substance disintegrates at a rate proportional to the amount present. If the rate constant is 1 percent per day, how long will it take for the amount to have reduced by half?

$\frac{dS}{dt} = -0.01 S \rightarrow S(t) = S_0 e^{-0.01t}$

where $S_0$ is the initial amount and $t$ is measured in days. $S/S_0 = 1/2 = \exp(-0.01t_*) \rightarrow \ln(1/2) = -0.01t_*$ or $t_* = 100 \ln 2$, about 69.3 days.

5. [10pts] Suppose you borrow $A_0$ euros at the rate of $r$ ('%/year) and interest is compounded continuously. (a) What is the effective Annual percentage rate? (b) If you pay back the money continuously at the constant rate of $p$ ('euros/week), what is the differential equation that determines the amount of money owed at time $t$? (c) Solve that equation.

$\frac{dA}{dt} = rA - p \rightarrow \frac{dA}{rA - p} = dt \rightarrow \ln(rA - p) = rt + C$.

We get the constant in terms of the initial amount borrowed by evaluating this at $t = 0$, yielding $C = \ln(rA_0 - p)$. Then, taking the exp of both sides:

$\frac{rA - p}{rA_0 - p} = e^{rt} \rightarrow A(t) = \frac{p}{r} + \left(A_0 - \frac{p}{r}\right)e^{rt}$.
Note that if \( A_0 - p/r > 0 \) the loan will never be paid back.

6. [10 pts] Sketch \( y = x^{1/17} \) and \( y = \ln x \) on the same plot. Calculate (a) \( \lim_{x \to 0^+} x^{1/17} \ln x \), (b) \( \lim_{x \to +\infty} x^{-1/17} \ln x \). Find a number \( M > 0 \) such that \( x^{1/17} > \ln x \) for all \( x > M \).

(a) \( \lim_{x \to 0^+} x^{1/m} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/m}} = \text{“0/0”} \). Use l’Hospital: \( \lim_{x \to 0^+} \frac{1/x}{(-1/m)(x^{-1/m-1})} = \lim_{x \to 0^+} -m x^{1/m} = 0, \forall m > 0 \).

(b) \( \lim_{x \to +\infty} x^{-1/17} \ln x = \lim_{x \to +\infty} \frac{\ln x}{x^{1/17}} = \text{“\(17/\infty\)”} \). Can use l’Hospital again: \( \lim_{x \to +\infty} 17 \frac{1/x}{x^{1/17} - 1} \) = \( \lim_{x \to +\infty} \frac{17}{x^{1/17}} \) = 0. SO \( x^{1/17} \) eventually beats the log. It sure does not look like that on the plot!

(c) Let’s compare \( x^{1/17} \) and \( \ln x \) further. How big should \( x \) be before \( x^{1/17} \) become bigger than \( \ln x \)? \( x \) will have to be really, really big, so maybe we should look at its \( \ln \), as the log of a big number is much smaller than that number (log increases slowly). Let \( y = \ln x \) then

\[
x^{1/17} \equiv e^{(\ln x)/17} > \ln x \implies \exp(y/17) > y.
\]

That’s more manageable. With a bit of fiddling (or Newton’s method) you can find that \( y \) must be bigger than 72.9189179, meaning that \( x \) must be bigger than \( e^{72.9189179} \approx 4.658910^{31} \). A pretty big number indeed.